ABOUT THE BOOK

This book deals about investigation of some common fixed point and coupled fixed point theorems in various spaces. New common coupled fixed point theorems for contractive inequalities using an auxiliary function which dominate the ordinary metric function are introduced. Some Presic type fixed point theorems for four maps in Fuzzy metric spaces were studied. C* - algebra valued fuzzy soft metric spaces also mentioned. Contractive conditions of integral type in dis located quasi b – metric spaces are used to study common fixed point theorems.

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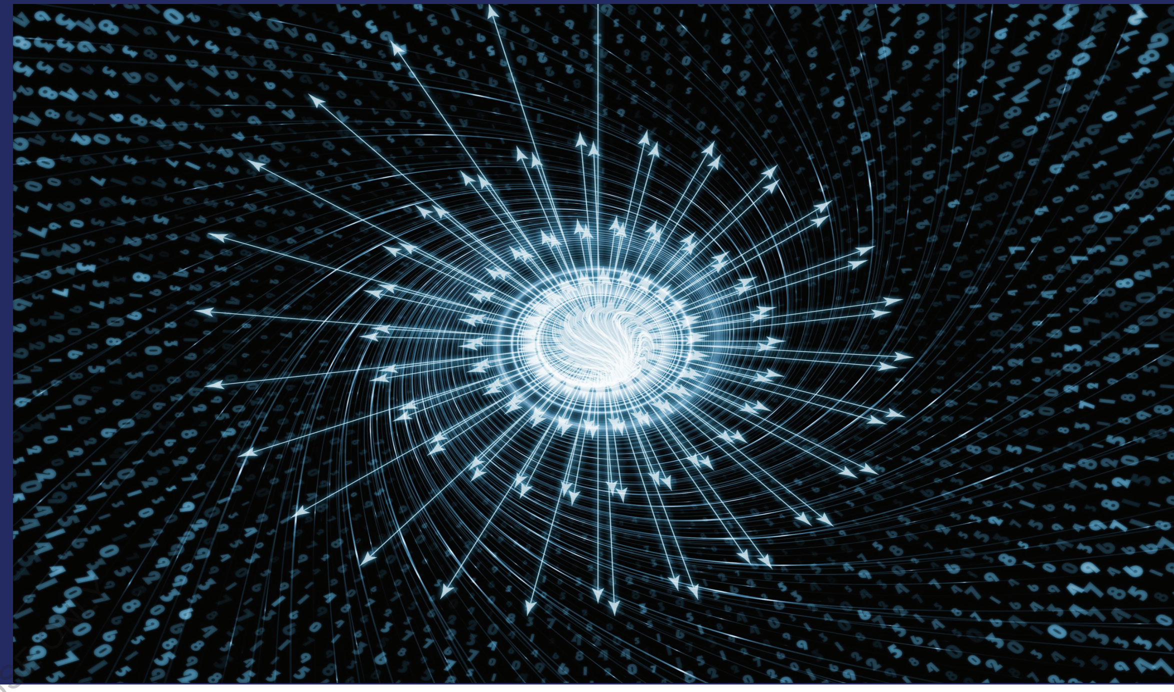
STUDY ON FIXED AND COUPLED FIXED POINTS

Dr. Mohammad Mustaq Ali



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CHAPTER-1

INTRODUCTION AND PRELIMINARIES

In recent times the study of fixed point theory has been gained an important role because of its wide applications in proving the existence and uniqueness of solutions of differential, integral, integro - differential and impulsive differential equations and in obtaining solutions of optimization problems, in Approximation theory and Non-linear Analysis. Further many fixed point theorems are used not only in various mathematical investigations but also problems in economics, game and computer theory.

In this chapter, we mention some known definitions, propositions and some main theorems in fixed point theory that are relevant to the content of this thesis.

Throughout this thesis, we denote $\mathbb{R}$ as the set of all real numbers, $\mathbb{R}^+$ as the set of all non - negative real numbers, $\mathbb{N}$ as the set of all natural numbers and $\mathbb{C}$ as the set of all complex numbers.

Suppose that X is a non-empty set and $T : X \to X$ is a self map on X. If there is an element $x \in X$ such that $Tx = x$, then x is called a fixed point of T in X.

Section 1.1 : BANACH FIXED POINT THEOREM FOR SELF MAPS

The fundamental work in fixed point theory is due to Banach (1922), which is famous as " Banach Contraction Principle ".

Theorem 1.1.1.(Banach Contraction Principle, [81]): Let (X, d) be a complete metric space and T be a self map on X and $0 \leq k < 1$ such that

$$d(Tx, Ty) \leq kd(x, y), \ \forall \ x, y \in X.$$

Then T has a unique fixed point in X. Further for any $x_0 \in X$, the sequence of iterates $\{T^n x_0\}$ is Cauchy and its limit is the unique fixed point of T.

Definition 1.1.2. Let X be a non-empty set and $T_1, T_2 : X \to X$ be given self maps on X.

1. If $T_1 x = T_2 x$ for some $x \in X$, then x is called a coincidence point of T_1 and T_2.

2. If $x = T_1 x = T_2 x$ for some $x \in X$, then x is called a common fixed point of T_1 and T_2.

3. (Jungck and Rhoades,[31]). If $T_1 T_2 x = T_2 T_1 x$ whenever there exists $x \in X$ such that $T_1 x = T_2 x$, then the pair (T_1, T_2) is said to be weakly compatible.

Now we give the basic definition of a partially ordered set as follows:

Definition 1.1.3. A partially ordered set is a set X and a binary relation $\preceq$ denoted by $(X, \preceq)$ such that, $\forall \ a, b, c \in X$

1. $a \preceq a$ (reflexivity),

2. $a \preceq b$ and $b \preceq a$ implies $a = b$ (anti - symmetry) and

3. $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ (transitivity).

Definition 1.1.4. Let $(X, \preceq)$ be a partially ordered set and $x, y \in X$. We say that x is comparable to y if either $x \preceq y$ or $y \preceq x$.

Section 1.2: G - METRIC SPACES

Dhage et al.[10,11,12,13] introduced the concept of D-metric spaces as generalization of ordinary metric functions and went on to present several fixed point results for single and multivalued mappings. Mustafa and Sims [113] and Naidu et al. [93, 94, 95] demonstrated that most of the claims concerning the fundamental topological structure of D-metric space are incorrect. Alternatively, Mustafa and Sims [113] introduced more appropriate notion of generalized metric space or a G - metric space and obtained robust topological structure for this space. Later Zead Mustafa, Hamed Obiedat and Fadi Awawdeh [116], Mustafa, Shatanawi and Bataineh [117], Mustafa and Sims [114], Shatanawi [107] and Renu Chugh, Tamanna Kadian, Anju Rani and B.E.Rhoades [21] obtained some fixed point theorems for a single map in G-metric spaces.

Definition 1.2.1(Mustafa et al.[113]): Let X be a nonempty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

$(G1)$: $G(x, y, z) = 0$ if $x = y = z$,

$(G2)$: $0 < G(x, y, z)$ for all $x, y \in X$ with $x \neq y$,

$(G3)$: $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

$(G4)$: $G(x, y, z) = G(x, z, y) = G(y, z, x) =$ .... symmetry in all three variables,

$(G5)$: $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a generalized metric or a G-metric on X and the pair (X, G) is called a G-metric space.

Clearly these properties are satisfied when $G(x, y, z)$ is the perimeter of the triangle at x, y and z in $\mathbb{R}^2$, further taking a in the interior of the triangle shows that (G_5) is best possible.

Definition 1.2.2 (Mustafa et al.[113]): Let (X, G) be a G-metric space in X. A point $x \in X$ is said to be limit of $\{x_n\}$ iff $\lim\limits_{n,m\to\infty} G(x, x_n, x_m) = 0$. In this case the sequence $\{x_n\}$ is said to be G-convergent to x.

Definition1.2.3 (Mustafa et al.[113]): Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence in X. $\{x_n\}$ is called G-Cauchy if and only if

$\lim\limits_{l,n,m\to\infty} G(x_l, x_n, x_m) = 0$.

(X, G) is called $G-$ complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 1.2.4(Mustafa et al.[113]): In a G-metric space (X, G), the following are equivalent.

1 The sequence $\{x_n\}$ is $G-$Cauchy.

2 For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.

Proposition 1.2.5(Mustafa et al.[113]): Let (X, G) be a G-metric spce. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Proposition 1.2.6(Mustafa et al.[113]): Let (X, G) be a G-metric space. Then for any $x, y, z, a \in X$, it follows that

(i) if $G(x, y, z) = 0$ then $x = y = z$,

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \leq 2\, G(x, x, y)$,

(iv) $G(x,y,z) \leq \frac{2}{3}\ [G(x,a,a) + G(y,a,a) + G(z,a,a)]$.

Proposition 1.2.7 (Mustafa et al.[113]): Let (X, G) be a G-metric spce. Then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$, the following are equivalent.

(i) $\{x_n\}$ is G-convergent to x,

(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Section 1.3 : PRESIC TYPE FIXED POINT THEOREMS

There are a number of generalizations of Banach contraction principle for multivalued mappings and hybrid pair of mappings for example (refer[9, 14, 32, 35, 44, 50, 105, 106]).

One such generalization is given by S.B.Presic [84] in 1965.

Let $f : X^k \to X$, where $k \geq 1$ is a positive integer. A point $x^* \in X$ is called a fixed point of f if $x^* = f(x^*, x^*, ...x^*)$. Consider the k-order non linear difference equation.

$$x_{n+1} = f(x_{n-k+1}, x_{n-k+2}, ...x_{k-1})\ for\ n = k-1, k, k+1. \qquad (A)$$

Equation (A) can be studied by means of fixed point theory in view of the fact that $x \in X$ is a solution of (A) if and only if x is a fixed point of f. One of the most important result in this direction is obtained by Presic [84] in the following way.

Theorem 1.3.1 (Presic et al.[84]): Let (X, d) be a complete metric space, k be a positive integer and $f : X^k \to X$ be a mapping satisfying

(1.2.1.1) $d(f(x_1, x_2, \cdots, x_k), f(x_2, x_3, \cdots, x_{k+1})) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1})$

for all $x_1, x_2, \cdots, x_k, x_{k+1} \in X$, where $q_i \geq 0$ and $\sum_{i=1}^{k} q_i < 1$. Then there exists a unique point $x \in X$ such that $f(x, x, ..., x) = x$. Moreover, if $x_1, x_2, \cdots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k} = f(x_n, x_{n+1}, \cdots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = f(lim x_n, lim x_n, ...lim x_n)$.

Later Ciric and Presic [51] generalized the above theorem as follows.

Theorem 1.3.2 (Ciric and Presic[51]): Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a mapping satisfying

$d(f(x_1, x_2, \cdots, x_k), f(x_2, x_3, \cdots, x_{k+1})) \leq \lambda \; max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}$

for all $x_1, x_2, \cdots, x_k, x_{k+1}$ in X and $\lambda \in [0, 1)$. Then there exists a point $x \in X$ such that $x = f(x, x, ..., x)$.

Moreover, if $x_1, x_2, \cdots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k} = T(x_n, x_{n+1}, \cdots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = f(lim x_n, lim x_n, ...lim x_n)$. If in addition, we suppose that on diagonal $\Delta \subset X^k$, $d(f(u, u, ..., u), f(v, v, ..., v)) < d(u, v)$ holds for $u, v \in X$ with $u \neq v$, then x is the unique fixed point satisfying $x = f(x, x, ..., x)$.

Recently Rao et al. [42, 45] obtained some Presic fixed point theorems for two and three maps in metric spaces. Now we give the following definition of Rao et al.[42]

Definition 1.3.3(Rao et al.[42]): Let X be a nonempty set, k a positive integer and $T : X^{2k} \to X$ and $f : X \to X$. The pair (f, T) is said to be $2k$-weakly compatible if $f(T(x, x, ..., x)) = T(fx, fx, ..., fx)$ whenever there exists $x \in X$ such that $fx = T(x, x, ..., x)$.

Using this definition, Rao et al.[42], proved the following theorem.

Theorem 1.3.4(Rao et al.[42]): Let (X, d) be a metric space and k be any positive integer. Let $S, T : X^{2k} \longrightarrow X$ and $f : X \longrightarrow X$ be mappings satisfying

(1.3.4.1) $d\begin{pmatrix} S(x_1, x_2, ..., x_{2k}), \\ T(x_2, x_3, ..., x_{2k+1}) \end{pmatrix} \leq \lambda \max\{d(fx_i, fx_{i+1}) : 1 \leq i \leq 2k\}$
$\forall\ x_1, x_2, ..., x_{2k}, x_{2k+1} \in X, where\ 0 \leq \lambda < 1.$

(1.3.4.2) $d\begin{pmatrix} S(y_1, y_2, ..., y_{2k}), \\ T(y_2, y_3, ..., y_{2k+1}) \end{pmatrix} \leq \lambda \max\{d(fy_i, fy_{i+1}) : 1 \leq i \leq 2k\}$
$\forall\ y_1, y_2, ..., y_{2k}, y_{2k+1} \in X, where\ 0 \leq \lambda < 1.$

(1.3.4.3) $d(S(u, u, ..., u), T(v, v, ..., v)) < d(fu, fv)\ \forall\ u, v \in X$ with $u \neq v$.

(1.3.4.4) Suppose that $f(X)$ is complete and either (f, S) or (f, T) is $2k$-weakly compatible pair.

Then there exists a unique point $p \in X$ such that $p = fp = S(p, p, .., p, p) = T(p, p, .., p, p)$.

Section 1.4 : COUPLED FIXED POINTS

Bhaskar and Lakshmikantham [101] introduced the concept of coupled fixed points and Lakshmikantham and Ciric [104] defined the common coupled fixed points. Abbas et al. [55] introduced the w-compatible mapping and proved some common coupled fixed point theorems in Cone metric spaces. Later several authors obtained coupled fixed and common coupled fixed point theorems in various spaces, (see for example [46,55,58,101,104]).

Definition 1.4.1(Bhaskar et al.[101]): Let X be a non-empty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.4.2([Lakshmikantham et al.104]): Let X be a non-empty set.

(i) An element $(x, y) \in X \times X$ is called coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

(ii) An element $(x, y) \in X \times X$ is called common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x) = x$ and $F(y, x) = g(y) = y$.

Definition 1.4.3(Abbas et al.[55]): Let X be a non-empty set. Let $S : X \times X \to X$ and $f : X \to X$ be mappings. Then the pair (S, f) is called w - compatible if $f(S(x, y)) = S(fx, fy)$ and $f(S(y, x)) = S(fx, fy)$ whenever there exist $x, y \in X$ with $f(x) = S(x, y)$ and $f(y) = S(y, x)$.

Section 1.5: FUZZY METRIC SPACES

The concept of fuzzy sets was introduced initially by L.Zadeh in 1965 [49]. George and Verramani[8] modified the concept of fuzzy topological spaces induced by fuzzy metric introduced by Grabeic[64] and proved the contraction principle in the settings of fuzzy metric spaces. Many authors(see for example[8, 40, 76, 84, 96]) have proved fixed and common fixed point theorems in fuzzy metric spaces.

Definition 1.5.1(Schweizer et al.[15]): A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

1. $*$ is associative and commutative,

2. $*$ is continuous,

3. $a * 1 = a$ for all $a \in [0, 1]$,

4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Two typical examples of a continuous t-norm are $a*b = ab$ and $a*b = \min\{a, b\}$.

Definition 1.5.2(George et al.[8]): A 3-tuple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(M_1) $M(x, y, t) > 0$,

(M_2) $M(x, y, t) = 1$ if and only if $x = y$,

(M_3) $M(x, y, t) = M(y, x, t)$,

(M_4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(M_5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the *open ball* $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by
$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$.
If $(X, M, *)$ is a fuzzy metric space, let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence in the sense of [8] if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$, for all $t > 0$ and each positive integer p. The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent.

Example 1.5.3. Let $X = [0, 1]$ and $a * b = ab$ for all $a, b \in [0, 1]$ and let M

be the fuzzy set on $X \times X \times (0, \infty)$ defined by

$M(x, y, t) = e^{-\frac{|x-y|}{t}}$ for all $t > 0$. Then $(X, M, *)$ is a fuzzy metric space.

Lemma 1.5.4(Grabiec et al.[62]: Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is non-decreasing with respect to t, for all $x, y \in X$.

Definition 1.5.5(Lopez et al.[40]): Let $(X, M, *)$ be a fuzzy metric space. Then M is said to be continuous on $X^2 \times (0, \infty)$

if $\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$, whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$.

i.e. $\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1$ and $\lim_{n\to\infty} M(x, y, t_n) = M(x, y, t)$.

Lemma1.5.6(Lopez et al.[40]): Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.

Section 1.6 : C^* - ALGEBRA VALUED FUZZY SOFT METRIC SPACES

In daily life, the problems in many fields deal with uncertain data and are not successfully modeled in classical mathematics. There are two types of mathematical tools to deal with uncertainties namely fuzzy set theory introduced by Zadeh [49] and the theory of soft sets initiated by Molodstov [23] which helps to solve problems in all areas. In [97] Thangaraj Beaula et al. defined fuzzy soft metric space in terms of fuzzy soft points and proved some results. On the other hand many authors proved so many results on fuzzy soft sets and fuzzy soft metric spaces (see [27, 97, 98, 100]).

In 2006, Ma et al. in [61] introduced a concept of C^*- algebra valued metric space and established some fixed and coupled fixed point results for mapping under contraction conditions in these spaces. for example, refer(see$[18, 33, 82, 99, 111]$).

Recently, R.P.Agarval et al.[79] initiate the concept of C*-algebra valued fuzzy soft metric spaces and proved some related fixed point results on this space (refer$[17, 79]$).

Throughout our discussion, U refers to an initial universe, E the set of all parameters for U and $P(\tilde{U})$ the set of all fuzzy set of U. (U, E) means the universal set U and parameter set E, $\tilde{C}$ refer to C^*-algebra.

The details on C^*-algebras are available in [30].

An algebra '$\tilde{C}$' together with a conjugate linear involution map $*: \tilde{C} \to \tilde{C}$, defined by $\tilde{a} \to \tilde{a}^*$ such that for all $\tilde{a}, \tilde{b} \in \tilde{C}$, we have $(\tilde{a}\tilde{b})^* = \tilde{b}^*\tilde{a}^*$ and $(\tilde{a}^*)^* = \tilde{a}$, is called a $\star$ - algebra.

Moreover, if $\tilde{C}$ an identity element $\tilde{I}_{\tilde{C}}$, then the pair $(\tilde{C}, \star)$ is called a unital $\star$ - algebra.

A unital $\star$ - algebra $(\tilde{C}, \star)$ together with a complete sub multiplicative norm satisfying $\tilde{a} = \tilde{a}^*$ for all $\tilde{a} \in \tilde{C}$ is called a Banach $\star$ - algebra.

A C^* - algebra is a Banach $\star$-algebra $(\tilde{C}, \star)$ such that $\tilde{a}^*\tilde{a} = \tilde{a}^2$ for all $\tilde{a} \in \tilde{C}$.

An element $\tilde{a} \in \tilde{C}$ is called a positive element if $\tilde{a} = \tilde{a}^*$ and $\sigma(\tilde{a}) \subset R(C)^*$ is set of non-negative fuzzy soft real numbers, where $\sigma(\tilde{a}) = \{\lambda \in R(C)^* : \lambda\tilde{I} - \tilde{a}$, is non-invertible$\}$. If $\tilde{a} \in \tilde{C}$ is positive, we write it as $\tilde{a} \geq \tilde{0}_{\tilde{C}}$.

Using positive elements, one can define partial ordering on $\tilde{C}$ as follows: $\tilde{a} \preceq \tilde{b}$ if and only if $\tilde{0}_{\tilde{C}} \preceq \tilde{b} - \tilde{a}$. Each positive element '$\tilde{a}$'of a C^*-algebra $\tilde{C}$ has a unique positive square root. Subsequently, $\tilde{C}$ will denote a unital C^*-algebra with the identity element $\tilde{I}_{\tilde{C}}$. Further, $\tilde{C}_+$ is the set $\{\tilde{a} \in \ \tilde{0}_{\tilde{C}} \preceq \tilde{a}\}$ of positive element of $\tilde{C}$.

A C^*-algebra valued Fuzzy soft metric space is defined in the following .

Definition 1.6.1 (Ravi et al.[79]): Let $C \subseteq E$ and $\tilde{E}$ be the absolute fuzzy soft set that is $F_E(e) = \tilde{1}$ for all $e \in E$. Let $\tilde{C}$ denote the C^*-algebra. The C^*-algebra valued fuzzy soft metric using fuzzy soft points is defined as a mapping $\tilde{d_{c^*}} : \tilde{E} \times \tilde{E} \to \tilde{C}$ satisfying the following conditions.

(M_0) $\tilde{0}_{\tilde{C}} \preceq \tilde{d}(F_{e_1}, F_{e_2})$, for all $F_{e_1}, F_{e_2} \in \tilde{E}$,

(M_1) $\tilde{d_{c^*}}(F_{e_1}, F_{e_2}) = \tilde{0}_{\tilde{C}} \Leftrightarrow F_{e_1} = F_{e_2}$,

(M_2) $\tilde{d_{c^*}}(F_{e_1}, F_{e_2}) = \tilde{d_{c^*}}(F_{e_2}, F_{e_1})$,

(M_3) $\tilde{d_{c^*}}(F_{e_1}, F_{e_3}) \preceq \tilde{d_{c^*}}(F_{e_1}, F_{e_2}) + \tilde{d_{c^*}}(F_{e_2}, F_{e_3})\ \forall\ F_{e_1}, F_{e_2}, F_{e_3} \in \tilde{E}$.

The fuzzy soft set $\tilde{E}$ with the C^*-algebra valued fuzzy soft metric $\tilde{d_{c^*}}$ is called the C^*-algebra valued fuzzy soft metric space. It is denoted by $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$. It is obvious that C^*-algebra valued fuzzy soft metric generalize the concept of fuzzy soft metric spaces, replacing the set of fuzzy soft real numbers by $\tilde{C}_+$.

Definition 1.6.2 (Ravi et al.[79]): A sequence $\{F_{e_n}\}$ in a C^*-algebra valued fuzzy soft metric space $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ is said to converges to $F_{e^{\shortmid}}$ in $\tilde{E}$ with respect to $\tilde{C}$, if $||\tilde{d_{c^*}}(F_{e_n}, F_{e^{\shortmid}})||_{\tilde{C}} \to \tilde{0}_{\tilde{C}}$ as $n \to \infty$ that is for every $\tilde{0}_{\tilde{C}} \prec \tilde{\epsilon}$ there exists $\tilde{0}_{\tilde{C}} \prec \tilde{\delta}$ and a positive integer $N = (\tilde{\epsilon})$ such that $||\tilde{d_{c^*}}(F_{e_n}, F_{e^{\shortmid}})|| < \tilde{\delta}$ implies that $||\mu^a_{F_{e_n}}(s) - \mu^a_{F_{e^{\shortmid}}}(s)|| < \tilde{\epsilon}$ whenever $n \geq N$. It is usually denoted as $\lim\limits_{n \to \infty} F_{e_n} = F_{e^{\shortmid}}$.

Definition 1.6.3 (Ravi et al.[79]): A sequence $\{F_{e_n}\}$ in a C^* - algebra valued fuzzy soft metric space $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ is said to be Cauchy sequence, if for every $\tilde{0}_{\tilde{C}} \prec \tilde{\epsilon}$ there exist $\tilde{0}_{\tilde{C}} \prec \tilde{\delta}$ and a positive integer $N = N(\tilde{\epsilon})$ such that $||\tilde{d_{c^*}}(F_{e_n}, F_{e_m})|| < \tilde{\delta}$ implies that $||\mu^a_{F_{e_n}}(s) - \mu^a_{F_{e_m}}(s)|| < \tilde{\epsilon}$ whenever $n, m \geq N$. That is $||\tilde{d_{c^*}}(F_{e_n}, F_{e_m})||_{\tilde{C}} \to \tilde{0}_{\tilde{C}}$ as $n, m \to \infty$.

Definition 1.6.4 (Ravi et al.[79]): A C^*-algebra valued fuzzy soft metric space $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ is said to be complete, if every Cauchy sequence in $\tilde{E}$ converges to some fuzzy soft point of $\tilde{E}$.

Example 1.6.5 (Ravi et al.[79]): Let $C \subseteq R$ and $E \subseteq R$, let $\tilde{E}$ be absolute fuzzy soft set, that is $\tilde{E}(e) = \tilde{1}$ for all $e \in E$, and $\tilde{C} = M_2(R(A)^*)$, define $\tilde{d_{c^*}} : \tilde{E} \times \tilde{E} \to \tilde{C}$ by

$$\tilde{d_{c^*}}(F_{e_1}, F_{e_2}) = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

where $i = \inf\{|\mu^a_{F_{e_1}}(s) - \mu^a_{F_{e_2}}(s)|/s \in C\}$ and $F_{e_1}, F_{e_2} \in \tilde{E}$. Then $\tilde{d_{c^*}}$ is a C^* - algebra valued fuzzy soft metric and $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ is a complete C^* - algebra valued fuzzy soft metric space by the completeness of $R(C)^*$.

Lemma 1.6.6(Ravi et al.[79]): Let $\tilde{C}$ be a C^*-algebra with the identity element $\tilde{I}_{\tilde{C}}$ and $\tilde{x}$ be a positive element of $\tilde{C}$. If $\tilde{a} \in \tilde{C}$ is such that $||\tilde{a}|| < 1$ then for $m < n$,

we have $\lim\limits_{n\to\infty} \sum_{k=m}^{n} (\tilde{a}^\star)^k \tilde{x} (\tilde{a})^k = \tilde{I}_{\tilde{C}} ||\tilde{(x)}^{\frac{1}{2}}||^2 \left(\frac{||\tilde{a}||^m}{1-||\tilde{a}||}\right)$....................(I)

and $\sum_{k=m}^{n} (\tilde{a}^\star)^k \tilde{x} (\tilde{a})^k \to \tilde{0}_{\tilde{C}}$ as $m \to \infty$................(II)

Lemma 1.6.7 (Ravi et al.[79]): Suppose that $\tilde{C}$ is a unital C^*-algebra with unit $\tilde{1}$.

(i) If $\tilde{a} \in \tilde{C}_+$ with $||\tilde{a}|| < \frac{1}{2}$ then $\tilde{I} - \tilde{a}$ is invertible and $||\tilde{a}(\tilde{I} - \tilde{a})^{-1}|| < 1$,

(ii) suppose that $\tilde{a}, \tilde{b} \in \tilde{C}$ with $\tilde{a}, \tilde{b} \succeq \tilde{0}_{\tilde{C}}$ and $\tilde{a}\tilde{b} = \tilde{b}\tilde{a}$ then $\tilde{a}\tilde{b} \succeq \tilde{0}_{\tilde{C}}$,

(iii) $\tilde{C}'$ we denote the set $\{\tilde{a} \in \tilde{C}/\tilde{a}\tilde{b} = \tilde{b}\tilde{a} \ \forall \ \tilde{b} \in \tilde{C}\}$. Let $\tilde{a} \in \tilde{C}'$, if $\tilde{b}, \tilde{c} \in \tilde{C}$

with $\tilde{b} \succeq \tilde{c} \succeq \tilde{0}$ and $\tilde{I} - \tilde{a} \in \tilde{C}'_+$ is an invertible operator, then $(\tilde{I} - \tilde{a})^{-1}\tilde{b} \succeq (\tilde{I} - \tilde{a})^{-1}\tilde{c}$.

Notices that in C^*-algebra, if $\tilde{0} \preceq \tilde{a}, \tilde{b}$ one can't conclude that $\tilde{0} \preceq \tilde{a}\tilde{b}$. Indeed, consider the C^*-algebra $M_2(R(C)^*)$ and set

$$\tilde{a} = \begin{bmatrix} F_{e_1}(a) & F_{e_2}(a) \\ F_{e_2}(a) & F_{e_1}(b) \end{bmatrix} = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$$

and $\tilde{b} = \begin{bmatrix} F_{e_1}(c) & F_{e_2}(c) \\ F_{e_2}(c) & F_{e_1}(d) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 0.5 & 0.6 \end{bmatrix}$

then clearly $\tilde{a} \succeq \tilde{0}$ and $\tilde{b} \succeq \tilde{0}$ but $\tilde{a}, \tilde{b} \in M_2(R(C)^*)_+$ while $\tilde{a}\tilde{b} \not\succeq \tilde{0}$.

Section 1.7: COMPLEX VALUED METRIC SPACES

Azam et al.[2] introduced the concept of a complex valued metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive type condition. Subsequently, Rouzkard and Imdad[29] established some common fixed point theorems for maps satisfying certain rational expressions in complex valued metric spaces to generalize the results of [2]. In the same way, Sintunavarat et al.[108, 109] obtained common fixed point results by replacing the constant of contractive condition to control functions. Recently, Sitthikul and Saejung [48] and Klin-eam and Suanoom [19] established some fixed point results by generalizing the contractive conditions in the context of complex valued metric spaces. Very recently, Ahmad et al.[38] obtained some new fixed point results for multi-valued mappings in the setting of complex valued metric spaces.

A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first co-ordinate is called $Re(z)$ and second co-ordinate is called $Im(z)$. Let $z_1, z_2 \in \mathbb{C}$.

Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$.

Thus $z_1 \precsim z_2$ if one of the following holds:

(1)$Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(2)$Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(3)$Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(4)$Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Definition 1.7.1 (Azam et al. [2]): Let X be a non empty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \precsim d(x, z) + d(z, y)$.

The pair (X, d) is called a complex valued metric space.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \preceq c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent to x and x is called the limit point of $\{x_n\}$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is called Cauchy sequence in(X, d). If every Cauchy sequence is convergent in (X, d) then (X, d) is called a complete complex valued metric space.

Lemma 1.7.2(Azam et al. [2]): Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma1.7.3(Azam et al. [2]): Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n, m \to \infty$.

Remark 1.7.4(Ahmad et al.[38]): Let (X, d) be a complex valued metric space and let CB(X) be a collection of nonempty closed subsets of X. Let $T : X \to CB(X)$ be a multi-valued map. For $x \in X$ and $A \in CB(X)$,
define $W_x(A) = \{d(x, a) : a \in A\}$.
Thus, for $x, y \in X$. $W_x(Ty) = \{d(x, u) : u \in Ty\}$.

Definition 1.7.5(Ahmad et al.[38]): Let (X, d) be a complex valued metric space. A nonempty subset A of X is called bounded from below if there exists some $z \in \mathbb{C}$ such that $z \precsim a$ for all $a \in A$.

Definition 1.7.6(Ahmad et al.[38]): Let (X, d) be a complex valued metric space. A multivalued mapping $F : X \to 2^C$ is called bounded from below if for each $x \in X$ there exists $z_x \in \mathbb{C}$ such that $z_x \precsim u$ for all $u \in Fx$.

Definition 1.7.7(Ahmad et al.[38]): Let (X, d) be a complex valued metric space. The multi-valued mapping $T : X \to CB(X)$ is said to have the lower bound property (l.b.Property) on (X, d) if for any $x \in X$, the multi-valued mapping $F_x : X \to 2^C$ defined by $F_x(y) = W_x(Ty)$ is bounded from below. That is for $x, y \in X$, there exists an element $l_x(Ty) \in C$ such that $l_x(Ty) \precsim u$, for all $u \in W_x(Ty)$, where $l_x(Ty)$ is called a lower bound of T associated with (x, y).

Definition 1.7.8(Ahmad et al.[38]): Let (X, d) be a complex valued metric space. The multivalued mapping $T : X \to CB(X)$ is said to have the greatest lower bound proerty (g.l.b.Property) on (X, d) if the greatest lower bound of $W_x(Ty)$ exists in $\mathbb{C}$ for all $x, y \in X$. We denote $d(x, Ty)$ by the g.l.b.Property of $W_x(Ty)$. That is $d(x, Ty) = inf\{d(x, u) : u \in Ty\}$.

Definition 1.7.9(Kamaran et al.[102]): Let $f : X \to X, S : X \to CB(X)$. f is said to be S-weakly commuting at $x \in X$ if $f^2x \in Sfx$.

Section 1.8 : COMPLEX VALUED S-METRIC SPACES

In 2011, Azam et al.[2] introduced the concept of a complex valued metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions. Later several authors proved fixed and common fixed point theorems in complex valued metric spaces, for example(refer[3, 29, 34, 46, 48, 54, 68, 87, 108]). On other hand the concept of S-metric spaces was introduced by S.Sedghi[91]. Later several authors proved fixed point results in S-metric spaces for example (refer[39, 47, 53, 71, 90, 92].

Recently Nabil et al.[70] introduced the concept of Complex valued S- metric spaces and proved common fixed point theorem in Complex valued S-metric spaces.

Definition 1.8.1(Sedghi et al.[91]): Let X be a non-empty set. A S-metric on X is a function $S : X^3 \longrightarrow \mathbb{R}^+$ that satisfies the following conditions for all $x, y, z, a \in X$.

$(S1)$ $S(x, y, z) = o$ if and only if $x = y = z$,

$(S2)$ $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called a S-metric space.

Definition 1.8.2(Nabil et al.[70]): Let X be a non-empty set. A complex valued S-metric on X is a function $S : X^3 \to \mathbb{C}$ that satisfies the following conditions, for all $x, y, z, a \in X$:

(i) $0 \precsim S(x,y,z)$,

(ii) $S(x,y,z) = 0$ if and only if $x = y = z$,

(iii) $S(x,y,z) \precsim S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called a complex valued S-metric space.

Example 1.8.3: Let $X = \mathbb{C}$. Define $S : \mathbb{C}^3 \to \mathbb{C}$ by:
$S(z_1, z_2, z_3) = [|Re(z_1) - Re(z_3)| + |Re(z_2) - Re(z_3)|] + i[|Im(z_1) - Im(z_3)| + |Im(z_2) - Im(z_3)|]$. Then (X,S) is a complex valued S-metric space.

Definition 1.8.4(Nabil et al.[70]): If (X,S) is called a complex valued S-metric space, then

(1) A sequence $\{x_n\}$ in X converges to x if and only if for all ϵ such that $0 \prec \epsilon \in \mathbb{C}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(x_n, x_n, x) \prec \epsilon$ and we denote this by $\lim\limits_{n\to\infty} x_n = x$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for all ϵ such that $0 \prec \epsilon \in \mathbb{C}$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $S(x_n, x_n, x_m) \prec \epsilon$.

(3) An S-metric space (X,S) is said to be complete if for every Cauchy sequence is convergent.

Lemma 1.8.5(Nabil et al.[70]): Let (X,S) be a complex valued S-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|S(x_n, x_n, x)| \to 0$ as $n \to \infty$.

Lemma 1.8.6(Nabil et al.[70]): Let (X,S) be a complex valued S-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|S(x_n, x_n, x_{n+m})| \to 0$ as $n \to \infty$ and $m \to \infty$.

Lemma 1.8.7(Nabil et al.[70]): Let (X, S) be a complex valued S-metric space. Then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.

Section 1.9: S_b - METRIC SPACES

In 2012, Sedghi et al.[91] introduced the notion of S-metric space and proved several results. On the other hand the concept of b-metric space was in troduced by Czerwik[8]. Recently Sedghi et al.[89] defined S_b-metric spces by using the concept of S and b-metric spaces and proved common fixed points of four maps in S_b-metric spaces. Later several authors proved fixed and coupled fixed point results in S_b-metric spaces for example (refer[43, 73., 88, 110]).

Definition 1.9.1 (Sedghi et al.[89]): Let X be a non-empty set and $b \geq 1$ be given real number. Suppose that a mapping $S_b : X^3 \to \mathbb{R}^+$ be a function satisfying the following properties :

(S_b1) $0 < S_b(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z$,

(S_b2) $S_b(x, y, z) = 0 \Leftrightarrow x = y = z$,

(S_b3) $S_b(x, y, z) \leq b(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a))$ for all $x, y, z, a \in X$.

Then the function S_b is called a S_b-metric on X and the pair (X, S_b) is called a S_b-metric space.

Remark 1.9.2 (Sedghi et al.[89]): It should be noted that, the class of S_b-metric spaces is effectively larger than that of S-metric spaces. Indeed each S-metric space is a S_b-metric space with $b = 1$.

Following example shows that a S_b-metric on X need not be a S-metric on X.

Example1.9.3(Sedghi et al.[89]): Let (X, S) be a S-metric space, and $S_*(x, y, z) = S_b(x, y, z)^p$, where $p > 1$ is a real number. Note that S_* is a S_b-metric with $b = 2^{2(p-1)}$. Also, (X, S_*) is not necessarily a S-metric space.

Definition 1.9.4(Sedghi et al.[89]): Let (X, S_b) be a S_b-metric space. Then, for $x \in X$, $r > 0$ we defined the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with center x and radius r as follows respectively:

$$B_S(x, r) = \{y \in X : S_b(y, y, x) < r\},$$
$$B_S[x, r] = \{y \in X : S_b(y, y, x) \leq r\}.$$

Lemma 1.9.5 (Sedghi et al.[89]): In a S_b-metric space, we have

$$S_b(x, x, y) \leq bS_b(y, y, x)$$

and

$$S_b(y, y, x) \leq bS_b(x, x, y).$$

Lemma 1.9.6(Sedghi et al.[89]): In a S_b-metric space, we have

$$S_b(x, x, z) \leq 2bS_b(x, x, y) + b^2 S_b(y, y, z)$$

Definition 1.9.7(Sedghi et al.[89]): If (X, S_b) be a S_b-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) S_b-Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S_b(x_n, x_n, x_m) < \epsilon$ for each $m, n \geq n_0$.

(2) S_b-convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S_b(x_n, x_n, x) < \epsilon$ or $S_b(x, , x, x_n) < \epsilon$ for all $n \geq n_0$ and it is denoted by $\lim_{n\to\infty} x_n = x$.

Definition 1.9.8(Sedghi et al.[89]): A S_b-metric space (X, S_b) is called complete if every S_b-Cauchy sequence is S_b-convergent in X.

Lemma 1.9.9 (Sedghi et al.[89]): If (X, S_b) be a S_b-metric space with $b \geq 1$ and suppose that $\{x_n\}$ is a S_b-convergent to x, then we have

(i) $\frac{1}{2b} S_b(y, x, x) \leq \lim_{n\to\infty} \inf S_b(y, y, x_n)$
$\leq \lim_{n\to\infty} \sup S_b(y, y, x_n) \leq 2b S_b(y, y, x)$ and

(ii) $\frac{1}{b^2} S_b(x, x, y) \leq \lim_{n\to\infty} \inf S_b(x_n, x_n, y)$
$\leq \lim_{n\to\infty} \sup S_b(x_n, x_n, y) \leq b^2 S_b(x, x, y)$ for all $y \in X$

In particular, if $x = y$, then we have $\lim_{n\to\infty} S_b(x_n, x_n, y) = 0$.

Section 1.10 : COMPLEX VALUED S_b - METRIC SPACES

Recently N.Priyobarta et al.[72] inspired by the concept of S_b-metric spaces introduced the concept of Complex valued S_b-metric spaces and proved some fixed point theorems.

Definition 1.10.1(Priyobarta et al.[72]): Let X be a non empty set and $b \geq 1$ be a given real number. Suppose that a mapping $S : X^3 \to \mathbb{C}$ satisfies

(CS_b1) $0 \prec S(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,

(CS_b2) $S(x, y, z) = 0 \Leftrightarrow x = y = z$,

(CS_b3) $S(x, x, y) = S(y, y, x)$,for all $x, y \in X$,

(CS_b4) $S(x, y, z) \precsim b(S(x, x, a) + S(y, y, a) + S(z, z, a))$ for all $x, y, z, a \in X$.

Then, S is called a complex valued S_b -metric and (X, S) is called a complex valued S_b-metric space.

Definition 1.10.2(Priyobarta et al.[72]): Let (X, S) be a complex valued S_b-metric space, let $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is a complex valued S_b-convergent to x if for every $a \in \mathbb{C}$ with $0 < a$, there exists $k \in \mathbb{C}$ such that $S(x_n, x_n, x) \prec a$ or $S(x, x, x_n) \prec a$ for all $n \gtrsim k$ and denoted by $\lim\limits_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ is called complex valued S_b Cauchy if for every $a \in \mathbb{C}$ with $0 < a$, there exists $k \in \mathbb{C}$ such that $S(x_n, x_n, x_m) \prec a$ for each $n, m \geq k$.

(iii) If every complex valued S_b-Cauchy sequence is complex valued S_b-convergent in (X, S), then (X, S) is said to be complex valued S_b complete.

Proposition 1.10.3(Priyobarta et al.[72]): Let (X, S) be a complex valued S_b-metric space and let $\{x_n\}$ be a sequence in X. Then (X, S) is complex valued S_b-convergent to x if and only if $|S(x_n, x_n, x)| \to 0$ as $n \to \infty$ or $|S(x, x, x_n)| \to 0$ as $n \to \infty$.

Theorem 1.10.4(Priyobarta et al.[72]): Let (X, S) be a complex valued S_b-metric space, then for a sequence $\{x_n\}$ in X and a point $x \in X$, the following are equvalent

(1) $\{x_n\}$ is a complex valued S_b convergent to x.

(2) $|S(x_n, x_n, x)| \to 0$ as $n \to \infty$.

Theorem 1.10.5(Priyobarta et al.[72]): Let (X, S) be a complex valued S_b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued S_b

Cauchy sequence if and only if $|S(x_n, x_m, x_l)| \to 0$ as $n, m, l \to \infty$.

Section 1.11 : α - ADMISSIBLE FUNCTION

Samet et al. [16] introduced the concept of α-admissible mappings and Salimi et al.[77] modified the concept of Samet et al. [16].

Definition 1.11.1(Samet et al.[16]): Let X be a non-empty set and T be a self-mapping on X and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. T is said to be α-admissible mapping if $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow d(Tx, Ty) \geq 1$.

Definition 1.11.2(Karpinar et al.[26]): Let X be a non-empty set and T be an α-admissible mapping on X. T is said to be a triangular α-admissible mapping if $x, y, z \in X$, $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$.

Definition 1.11.3(Salimi et al.[77]): Let X be a non-empty set and T be a self-mapping on X and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. Then T is an α-admissible mapping with respect to η if $x, y \in X, \alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty)$.

Definition 1.11.4(Hussain et al.[69]): Let Ψ be the family of non-decrasing functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum\limits_{n=1}^{\infty} \psi^n(t) < t$ for each $t > 0$.

Section 1.12: DISLOCATED QUASI b-METRIC SPACES

Hitzler [75] and Hitzler and Seda [74] introduced the notion of dislocated metric spaces and generalized the celebrated Banach contraction principle in such spaces. Zeyada et al. [28] initiated the concept of dislocated quasi metric spaces and generalized the results of Hitzler and Seda[74] in dislocated quasi metric spaces. The notion of b-metric spaces was introduced by Czerwic [85] in connection with some problems concerning with the convergence of non measurable functions with respect to measure. In the year 2015, Klin-eam and Suanoom [20] introduced the concept of dislocated quasi b-metric spaces based on the concepts of b-metric spaces [85] and quasi b-metric spaces [63] and provided some fixed point theorems by using cyclic contractions. Later several authors worked on dislocated quasi b-metric spaces and obtained fixed and common fixed points using various contraction conditions for single map and two maps.

Definition 1.12.1(Klin-eam et al.[20]): Let X be a non-empty set and $k \geq 1$ be a real number then a mapping $d : X \times X \to \mathbb{R}^+$ is called dislocated quasi b-metric if $\forall\ x, y, z \in X$

$(d1)$ $d(x, y) = d(y, x) = 0$ implies that $x = y$,

$(d2)$ $d(x, y) \leq k[d(x, z) + d(z, y)]$.

The pair (X, d) is called dislocated quasi b-metric space.

Definition 1.12.2(Klin-eam et al.[20]): A sequence $\{x_n\}$ is called dislocated quasi b-convergent in (X, d) if $\lim\limits_{n\to\infty} d(x_n, x) = 0 = \lim\limits_{n\to\infty} d(x, x_n)$. Then x is called the dislocated quasi b-limit of the sequence $\{x_n\}$.

Definition 1.12.3(Klin-eam et al.[20]): A sequence $\{x_n\}$ in dislocated quasi b-metric space (X, d) is called Cauchy squence if $\lim\limits_{m,n\to\infty} d(x_m, x_n) = 0 = \lim\limits_{m,n\to\infty} d(x_n, x_m)$.

Definition 1.12.4(Klin-eam et al.[20]): A dislocated quasi b-metric space (X, d) is said to be complete if every Cauchy sequence in X convergent to a point of X.

Lemma 1.12.5(Klin-eam et al.[20]): Let(X, d) be a dislocated quasi b-metric space and $\{x_n\}$ be dislocated quasi b-convergent to $x \in X$ and $y \in X$ be arbitrary. Then $\frac{1}{k}d(x,y) \leq \lim_{n\to\infty} \ inf \ d(x_n,y) \ \leq \lim_{n\to\infty} \ sup \ d(x_n,y) \ \leq kd(x,y)$ and

$\frac{1}{k}d(y,x) \leq \lim_{n\to\infty} \ inf \ d(y,x_n) \leq \lim_{n\to\infty} \ sup \ d(y,x_n) \leq kd(y,x)$.

Note 1.12.6: $\frac{1}{2k}d(x,y) \leq \max\{d(x,z), d(z,y)\}$ for all $x, y, z \in X$.

SYNOPSIS OF THE THESIS

This thesis is divided into seven chapters.

Chapter 1: Introduction and Preliminaries

In this Chapter we present some known basic notions like fixed and coupled fixed points, concepts regarding to α - admissible maps, Presic type and Suzuki type fixed point theorems in metric, G - metric, complex valued metric, fuzzy metric, C^*-algebra valued fuzzy soft metric, complex valued S-metric, S_b-metric, complex valued S_b-metric and dislocated quasi b-metric spaces and as well as contents of the thesis.

Chapter 2: Common Fixed and Coincidence Point Theorems in Some Spaces

We divide this chapter into three sections namely, Section 2.1, Section 2.2 and Section 2.3.

In Section 2.1, first we prove a common fixed point theorem for three expansive mappings. Our result generalizes the theorems of Zead et al.[115]. Also we prove another theorem for two jungck type expansive mappings. And we obtain corollary for single map.

In Section 2.2, we introduce the definition of jointly $2k$-weakly compatible pairs of maps. We obtain a Presic type fixed point theorem for two pairs of jointly 2k-weakly compatible maps in fuzzy metric spaces. We obtain two corollaries for three and two maps respectively which are slight variations of theorems of Rao et al.[42,45].Our main result extends the theorem of Murthy et al.[76]. We also give an example to illustrate our main theorem.

In Section 2.3, we obtain a coincidence point theorem for two pairs of hybrid mappings in complex valued metric spaces. Our result generalize the theorem of Azam et al.[4].

Chapter 3: Coupled and Coincidence Point Theorems in C^*-Algebra Valued Fuzzy Soft Metric Spaces.

We divide this chapter into two sections namely, Section 3.1 and Section 3.2.

In Section 3.1, we establish the existence and uniqueness of common coupled fixed point results for three mappings in C^*-algebra valued fuzzy soft metric spaces. Moreover, we give an illustration which presents the applicability of the achieved results. Also we provided application to Integral Equations.

In Section 3.2, we obtain a coincidence point theorem for a hybrid pair of single valued and multivalued mappings in complete C^*-algebra valued fuzzy soft metric spaces. An example is also given to validate our results.

Chapter 4: Unique Common Fixed Point Theorem for Four maps in Complex valued S-metric Spaces.

In this Chapter, we prove a common fixed point theorem for four maps satisfying more general contractive condition using 7 functions in Complex valued S-metric spaces. We also provide an example to illustrate our result.

Our result generalize the theorem of Naval Singh et al.[68].

Chapter 5: Common And Coupled Fixed Point Teorems in S_b-Metric Spaces.

We divide this chapter into two sections namely, Section 5.1 and Section 5.2.

In Section 5.1, we obtain a uniquene common fixed point theorem for two weakly compatible pairs of mappings satisfing a contractive condition in Complex valued S_b-metric spaces. we also provide an example to illustrate our theorem. Our result generalize the theorem of N.Priyobarta et al.[72].

In Section 5.2, we obtain Suzuki type common coupled fixed point theorems in S_b metric spaces for four maps and single map respectively. We also furnish an example which supports our main result. Our result generalize the theorem of Sedghi et al.[89].

Chapter 6: A New Common Coupled Fixed Point Result For Contractive Maps Involving Dominating Functions

In this chapter we extend the Salimi et al.[77] Defintion from single map to jungck type maps of which one is a coupled map. Mainly we establish a new common coupled fixed point theorem for contractive inequalities using auxiliary function which dominate the ordinary metric function for two maps. Also obtain a common fixed point for four maps. Our result generalize the theorem of N.Hussain et al.[69].

Chapter 7: Unique Common Fixed Point Theorem of Integral Type Contraction For Four Maps In Dislocated Quasi b-Metric Spaces

In this Chapter, we prove two unique common fixed point theorems using contractive condition of integral type in dislocated quasi b-metric spaces. In the first theorem, we used the continuities of all four mappings and commutativity of two pair of maps. In the second theorem, we replaced the commutativity and continuity of maps in Theorem 7.4. by weakly compatible pairs and completeness of one of the range set of maps.

Our result extends the theorem of M.U.Rahman et al.[67]. We also give two examples to support our theorems.

After Chapter 7, we give a list of references used for the preparation of this thesis.

CHAPTER 2

COMMON FIXED AND COINCIDENCE POINT THEOREMS IN SOME SPACES

We divided Chapter 2 into three sections, namely, Section 2.1., Section 2.2. and Section 2.3. The main aim of the Chapter is to prove common fixed point theorem in G - metric spaces, a unique common fixed point theorem for four mappings satisfying Presic type condition in fuzzy metric spaces and concidence point theorem for two pairs of hybrid mappings in complex valued metric spaces.

SECTION 2.1: COMMON FIXED POINT THEOREM FOR EXPANSIVE MAPPINGS IN G-METRIC SPACES

Recently Zead et al.[115] proved the following theorems.

Theorem 2.1.1.(Zead et al.[115]): Let (X, G) be a complete G-metric space. If there exists a constant $a > 1$ and a surjective mapping T on X, such that for all $x, y, z \in X$

$$(i)\ \ G(Tx, Ty, Tz) \geq aG(x, y, z).$$

Then T has a fixed point.

Theorem 2.1.2.(Zead et al.[115]): Let (X, G) be a complete G-metric space, $T : X \to X$ an onto and continuous mapping satisfying the following condition for all $x \in X$

$$(i)\ \ G(T(x), T^2(x), T^3(x)) \geq aG(x, Tx, T^2x)$$

where $a > 1$. Then T has a fixed point.

In this section, we obtain a common fixed point theorem for three expansive mappings and a unique common fixed point theorem for two Jungck type expansive mappings in G-metric spaces. Our main theorem generalise the Theorem 2.1.1 and Theorem 2.1.2.

Now we give our Main Theorem.

Theorem 2.1.3. Let (X, G) be a complete G- metric space. If there exist a constant $q > 1$ and surjective mappings A, B and C on X such that

$$G(Ax, By, Cz) \geq q \max \left\{ \begin{array}{l} G(x, y, z), G(x, Ax, Cz), \\ G(y, By, Ax), G(z, Cz, By) \end{array} \right\}$$

for all $x, y, z \in X$, then

(a) A or B or C has a fixed point in X,

(or)

(b) A, B and C has a unique common fixed point in X.

Proof: Let $x_0 \in X$, there exist $x_1, x_2, x_3 \in X$ such that

$x_0 = Ax_1, x_1 = Bx_2, x_2 = Cx_3$.

By induction we have

$x_{3n} = Ax_{3n+1}, x_{3n+1} = Bx_{3n+2}, x_{3n+2} = Cx_{3n+3}, n = 0, 1, 2...$.

If $x_{3n+1} = x_{3n}$ then $Ax = x$ where $x = x_{3n}$.

If $x_{3n+2} = x_{3n+1}$ then $Bx = x$ where $x = x_{3n+1}$.

If $x_{3n+3} = x_{3n+2}$ then $Cx = x$ where $x = x_{3n+2}$.

Assume that $x_n \neq x_{n+1}$ for all n.

Denote $d_n = G(x_n, x_{n+1}, x_{n+2})$.

$$\begin{aligned} d_{3n-1} &= G(x_{3n-1}, x_{3n}, x_{3n+1}) \\ &= G(Cx_{3n}, Ax_{3n+1}, Bx_{3n+2}) \\ &\geq q \max \left\{ \begin{array}{l} G(x_{3n+1}, x_{3n+2}, x_{3n}), G(x_{3n+1}, x_{3n}, x_{3n-1}), \\ G(x_{3n+2}, x_{3n+1}, x_{3n}), G(x_{3n}, x_{3n-1}, x_{3n+1}) \end{array} \right\} \\ &= q \max \{d_{3n}, d_{3n-1}, d_{3n}, d_{3n-1}\}. \end{aligned}$$

Thus we have $d_{3n-1} \geq q d_{3n}$ so that

$$d_{3n} \leq k d_{3n-1} \quad \text{where} \quad k = \frac{1}{q} < 1 \tag{1}$$

$$\begin{aligned} d_{3n} &= G(x_{3n}, x_{3n+1}, x_{3n+2}) \\ &= G(Ax_{3n+1}, Bx_{3n+2}, Cx_{3n+3}) \\ &\geq q \max \left\{ \begin{array}{l} G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n+1}, x_{3n}, x_{3n+2}), \\ G(x_{3n+2}, x_{3n+1}, x_{3n}), G(x_{3n+3}, x_{3n+2}, x_{3n+1}) \end{array} \right\} \\ &= q \max \{d_{3n+1}, d_{3n}, d_{3n}, d_{3n+1}\}. \end{aligned}$$

Thus we have $d_{3n} \geq q d_{3n+1}$ so that $d_{3n+1} \leq k d_{3n}$ (2)

$$\begin{aligned} d_{3n+1} &= G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ &= G(Bx_{3n+2}, Cx_{3n+3}, Ax_{3n+4}) \\ &\geq q \max \left\{ \begin{array}{l} G(x_{3n+4}, x_{3n+2}, x_{3n+3}), G(x_{3n+4}, x_{3n+3}, x_{3n+2}) \\ G(x_{3n+2}, x_{3n+1}, x_{3n+3}), G(x_{3n+3}, x_{3n+2}, x_{3n+1}) \end{array} \right\} \\ &= q \max \{d_{3n+2}, d_{3n+2}, d_{3n+1}, d_{3n+1}\}. \end{aligned}$$

Thus we have $d_{3n+1} \geq q d_{3n+2}$ so that $d_{3n+2} \leq k d_{3n+1}$ (3)

From (1), (2) and (3) we have $d_n \leq k d_{n-1}, n = 1, 2, 3....$

From (G_3) we have

$$\begin{aligned} G(x_n, x_n, x_{n+1}) &\le G(x_n, x_{n+1}, x_{n+2}) \\ &\le kG(x_{n-1}, x_n, x_{n+1}) \\ &\le k^2 G(x_{n-2}, x_{n-1}, x_n) \\ &: \\ &: \\ &\le k^n G(x_0, x_1, x_2). \end{aligned}$$

Now using (G_5), for $m > n$

$$\begin{aligned} &G(x_n, x_n, x_m) \\ &\le G(x_n, x_n, x_{n+1}) + G(x_{n+1}, x_{n+1}, x_{n+2}) + G(x_{n+2}, x_{n+2}, x_{n+3}) + ... + G(x_{m-1}, x_{m-1}, x_m) \\ &\le (k^n + k^{n+1} + k^{n+2} + ... + k^{m-1})\, G(x_0, x_1, x_2) \\ &\le \tfrac{k^n}{1-k} G(x_0, x_1, x_2) \\ &\to 0 \text{ as } n \to \infty\ m \to \infty. \end{aligned}$$

Hence $\{x_n\}$ is G-Cauchy. Since (X, G) is complete, there exists $p \in X$ such that $\{x_n\}$ is G-convergent to p.

Now

$$\begin{aligned} G(Ap, x_{3n+1}, x_{3n+2}) &= G(Ap, Bx_{3n+2}, Cx_{3n+3}) \\ &\ge q \max \left\{ \begin{array}{l} G(p, x_{3n+2}, x_{3n+3}), G(p, Ap, x_{3n+2}), \\ G(x_{3n+2}, x_{3n+1}, Ap), G(x_{3n+3}, x_{3n+2}, x_{3n+1}) \end{array} \right\} \end{aligned}$$

letting $n \to \infty$, we get

$G(Ap, p, p) \ge q\ \max\{0, G(p, Ap, p), G(p, p, Ap), 0\}$.

Thus $G(Ap, p, p) = 0$ so that $Ap = p$.

$$\begin{aligned} G(x_{3n}, Bp, x_{3n+2}) &= G(Ax_{3n+1}, Bp, Cx_{3n+3}) \\ &\ge q\ \max \left\{ \begin{array}{l} G(x_{3n+1}, p, x_{3n+3}), G(x_{3n+1}, x_{3n}, x_{3n+2}), \\ G(p, Bp, x_{3n}), G(x_{3n+3}, x_{3n+2}, Bp) \end{array} \right\} \end{aligned}$$

letting $n \to \infty$ we get

$G(p, Bp, p) \geq q\ max\{0, 0, G(p, Bp, p), G(p, Bp, p)\}$.

Thus $G(p, Bp, p) = 0$ so that $Bp = p$.

$$\begin{aligned} G(x_{3n}, x_{3n+1}, Cp) &= G(Ax_{3n+1}, Bx_{3n+2}, Cp) \\ &\geq q \max \left\{ \begin{array}{l} G(x_{3n+1}, x_{3n+2}, p), G(x_{3n+1}, x_{3n}, Cp), \\ G(x_{3n+2}, x_{3n+1}, x_{3n}), G(p, Cp, x_{3n+1}) \end{array} \right\} \end{aligned}$$

letting $n \to \infty$ we get

$G(p, p, Cp) \geq q\ max\{0, G(p, p, Cp), 0, G(p, Cp, p)\}$.

Thus $G(p, p, Cp) = 0$ so that $Cp = p$.

Thus p is a common fixed point of A, B and C.

Now consider

$$\begin{aligned} G(p, p, p') &= G(Ap, Bp, Cp') \\ &\geq q\ \ \max\{G(p, p, p'), G(p, p, p'), 0, G(p', p', p)\} \\ &\geq q\ \ \max\left\{G(p, p, p'), \tfrac{1}{2}G(p, p, p')\right\}\ \ since\ G(p, p, p') \leq 2G(p', p', p) \\ &= q\ G(p, , p, p'). \end{aligned}$$

Hence $p' = p$.

Thus p is a unique common fixed point of A, B and C.

Corollary 2.1.4. Let (X, G) be a complete G- metric space. If there exist a constant $q > 1$ and surjective mapping T on X such that

$$G(Tx, Ty, Tz) \geq q \max \left\{ \begin{array}{l} G(x, y, z), G(x, Tx, Tz), \\ G(y, Ty, Tx), G(z, Tz, Ty) \end{array} \right\}$$

for all $x, y, z \in X$, then T has a unique fixed point in X.

Proof: Let $x_0 \in X$. There exists a sequence $\{x_n\}$ in X such that

$x_n = Tx_{n+1}, n = 0, 1, 2......$

If $x_n = x_{n+1}$ for some n then $Tx = x$, where $x = x_{n+1}$.

Assume that $x_n \neq x_{n+1}$ for all n.

The rest of the proof follows as in Theorem 2.1.3

Theorem 2.1.5. Let (X, G) be a G- metric space and $A, f : X \to X$ be satisfying

(2.1.5.1)

$$G(Ax, Ay, Az) \geq q \max \left\{ \begin{array}{l} G(fx, fy, fz), G(fx, Ax, fz) \\ G(fy, Ay, fx), G(fz, Az, fy) \end{array} \right\}$$

for all $x, y, z \in X$,where $q > 1$,

(2.1.5.2) $f(X) \subseteq A(X)$ and $f(X)$ is a G-complete sub space of X and

(2.1.5.3) the pair (A, f) is weakly compatible .

Then A and f have a unique common fixed point .

Proof: Let $x_0 \in X$. From (2.1.5.2), there exists $x_1 \in X$ such that

$fx_0 = Ax_1 = y_1$, say.

Inductively, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$fx_n = Ax_n = y_n, n = 1, 2, 3, ...$

Case(i): Suppose $y_n = y_{n+1}$ for some n. Then $fx_{n-1} = Ax_{n-1}$.

Thus $fp = Ap$ where $p = x_{n-1}$. Since (A, p) is weakly compatible,

we have $f^2p = f(fp) = f(Ap) = Afp = A^2p$.

$$\begin{aligned} G(A^2p, Ap, Ap) &\geq q \max \left\{ \begin{array}{l} G(fAp, fp, fp), G(fAp, AAp, fp), \\ G(fp, Ap, fAp), G(fp, Ap, fp) \end{array} \right\} \\ &= q \max \left\{ \begin{array}{l} G(A^2p, Ap, Ap), G(A^2p, A^2p, Ap), \\ G(Ap, Ap, A^2p), 0 \end{array} \right\} \\ &\geq qG(A^2p, A^2p, Ap), \end{aligned}$$

and similarly we get $G(A^2p, A^2p, Ap) \geq q^2 \, G(A^2p, Ap, Ap)$,

so $G(A^2p, Ap, Ap) \geq q^2 \, G(A^2p, A^2p, Ap)$ which is a contradiction.

Hence $A^2p = Ap$. Then $fAp = A^2p = Ap$.

Ap is a common fixed point of f and A.

Case(ii): Assume that $y_n \neq y_{n+1}$ for all n

$$\begin{aligned} G(y_{n-1}, y_{n-1}, y_n) &= G(Ax_{n-1}, Ax_{n-1}, Ax_n) \\ &\geq q \max \left\{ \begin{array}{l} G(y_n, y_n, y_{n+1}), G(y_n, y_{n-1}, y_{n+1}), \\ G(y_n, y_{n-1}, y_n), G(y_{n+1}, y_n, y_n) \end{array} \right\} \\ &\geq q \max \left\{ \begin{array}{l} G(y_n, y_n, y_{n+1}), G(y_{n-1}, y_{n-1}, y_n), \\ \frac{1}{2} G(y_{n-1}, y_{n-1}, y_n), G(y_n, y_n, y_{n+1}) \end{array} \right\}, \end{aligned}$$

since $G(y_{n+1}, y_{n-1}, y_n) = G(y_{n-1}, y_n, y_{n+1})$ and

$G(y_{n-1}, y_{n-1}, y_n) \leq 2 \, G(y_{n-1}, y_n, y_{n+1})$.

Thus $G(y_{n-1}, y_{n-1}, y_n \geq q \, G(y_n, y_n, y_{n+1})$.

Hence

$$\begin{aligned} G(y_n, y_n, y_{n+1}) &\leq kG(y_{n-1}, y_{n-1}, y_n) \ where \ k = \tfrac{1}{q} < 1 \\ &\leq k^2 G(y_{n-2}, y_{n-2}, y_{n-1}) \\ &\leq k^3 G(y_{n-3}, y_{n-3}, y_{n-2}) \\ &: \\ &: \\ &\leq k^n G(y_0, y_0, y_1). \end{aligned}$$

Now using (G_5), for $m < n$ we have

$$\begin{aligned} G(y_n, y_n, y_m) &\leq G(y_n, y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_{n+2}) + \ldots\ldots + G(y_{m-1}, y_{m-1}, y_n) \\ &\leq (k^n + k^{n+1} + \ldots + k^{m-1}) G(y_0, y_0, y_1) \\ &\leq \tfrac{k^n}{1-k} G(y_0, y_0, y_1) \\ &\to 0 \ as \ n \to \infty, m \to \infty. \end{aligned}$$

Hence $\{y_n\}$ is a G-Cauchy.

Since $f(X)$ is G-complete, there exists $p, t \in X$ such that $y_n \to p = ft$.

$$\begin{aligned} G(At, y_n, y_n) &= G(At, Ax_n, Ax_n) \\ &\geq q \max \left\{ \begin{array}{l} G(p, y_{n+1}, y_{n+1}), G(p, At, y_{n+1}), \\ G(y_{n+1}, y_n, p), G(y_{n+1}, y_n, y_n) \end{array} \right\} \end{aligned}$$

letting $n \to \infty$, we get

$G(At, p, p) \geq q\ G(p, At, p)$. Thus $At = p$. Hence $ft = At$.

As in case(i), $ft = At = p$ is common fixed point of f and A.

Uniqueness: Suppose p' is another common fixed point of A and f.

$$\begin{aligned} G(p, p, p') &= G(Ap, Ap, Ap') \\ &\geq q \max \left\{ \begin{array}{l} G(p, p, p'), G(p, p, p'), \\ 0, G(p, p, p') \end{array} \right\} \\ &\geq q \max \left\{ G(p, p, p'), \tfrac{1}{2} G(p, p, p') \right\} \\ &= q\ G(p, p, p'). \\ Hence\ p' &= p. \end{aligned}$$

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SECTION 2.2: A UNIQUE COMMON FIXED POINT THEOREM FOR FOUR MAPPINGS SATISFYING PRESIC TYPE CONDITION IN FUZZY METRIC SPACES

In this section, we obtain a Presic type common fixed point theorem for four maps in Fuzzy metric spaces. We also present one example to illustrate our main theorem. Further, we obtain two more corollaries.

In 2013 Murthy and Rashmi [76] defined the following function.

Definition 2.2.1(Murthy et al.[76]): Let $\phi : [0,1]^k \to [0,1]$ be such that

(2.2.1.1) ϕ is increasing and continuous function in each variable,

(2.2.1.2) $\phi(t,t,t,...,t) \geq t$ for all $t \in [0,1]$.

Using this function, Murthy and Rashmi[76] extended the Theorem 1.3.4(Ch-1) to fuzzy metric spaces as follows.

Theorem 2.2.2(Murthy et al.[76]): Let $(X, M, *)$ be a fuzzy metric space and $S, T : X^{2k} \longrightarrow X$ and $f : X \longrightarrow X$ be mappings satisfying for each positive integer k, $0 < q < \frac{1}{2}$ and $t \in \mathbb{R}^+$

(2.2.2.1) $M(S(x_1, x_2, ..., x_{2k}), T(x_2, ..., x_{2k}, x_{2k+1}), qt)$
$\geq \phi(M(fx_1, fx_2, t), ..., M(fx_{2k}, fx_{2k+1}, t)$ for all $x_1, x_2, ...x_{2k+1} \in X$,

(2.2.2.2) $M(T(y_1, y_2, ..., y_{2k}), S(y_2, y_3, ..., y_{2k+1}), qt)$
$\geq \phi(M(fy_1, fy_2, t), ..., M(fy_{2k}, fy_{2k+1}, t)$ for all $y_1, y_2, ...y_{2k+1} \in X$,

(2.2.2.3) $M(S(u, u, ..., u), T(v, v, ..., v), qt) > M(fu, fv, t)$ for all $u, v \in X$ with $u \neq v$.

Suppose that $f(X)$ is complete and either (f, S) or (f, T) is $2k$-weakly compatible pair.

Then there exists a unique $p \in X$ such that

$p = fp = S(p, p, ..., p) = T(p, p, ..., p)$.

Now we state the condition (A): $\lim_{t \to \infty} M(x, y, t) = 1$ for all $x, y \in X$.

We observe that in the proof of Theorem 2.2.2 the authors Murthy and Rashmi[76] inherently used the condition(A).

Now we introduce the definition of Jointly $2k$ weakly compatible pairs as follows.

Definition 2.2.3. Let X be a nonempty set, k a positive integer and $S, T : X^{2k} \to X$ and $f, g : X \to X$. The pair $(f, S)$and (g, T) are said to be jointly $2k$-weakly compatible if $f(S(x, x, ..., x) = S(fx.fx, ..., fx)$ and $g(T(x, x, ..., x)) = T(gx, gx, ..., gx)$ whenever there exists $x \in X$ such that $fx = S(x, x, ..., x)$ and $gx = T(x, x, ..., x)$.

Now we extend the Theorem 2.2.2 for four maps as follows using some different conditions.

Throughout this section assume ϕ as in Definition 2.2.1.

Theorem 2.2.4. Let $(X, M, *)$ be a fuzzy metric space with the condition (A), k a positive integer and $S, T : X^{2k} \longrightarrow X$ and $f, g : X \longrightarrow X$ be mappings satisfying:

(2.2.4.1) $S(X^{2k}) \subseteq g(X), T(X^{2k}) \subseteq f(X)$,

(2.2.4.2)

$$M(S(x_1, x_2, ..., x_{2k}), T(y_1, y_2, ..., y_{2k}), qt) \geq \phi \begin{pmatrix} M(gx_1, fy_1, t), M(fx_2, gy_2, t), \\ M(gx_3, fy_3, t), M(fx_4, gy_4, t), \\ \vdots \\ M(gx_{2k-1}, fy_{2k-1}, t), M(fx_{2k}, gy_{2k}, t) \end{pmatrix}$$

$\forall x_1, x_2, ..., x_{2k}, y_1, y_2, .., y_{2k} \in X, \forall t > 0, 0 < q < 1,$

(2.2.4.3) (f, S) and (g, T) are jointly $2k$-weakly compatible pairs.

(2.2.4.4) Suppose $z = fu = gu$ for some $u \in X$ whenever there exists a sequence $\{y_{2k+n}\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} y_{2k+n} = z \in X$.

Then z is the unique point in X such that $z = fz = gz = S(z, z, .., z, z) = T(z, z, ..., z, z)$.

Proof: Suppose $x_1, x_2, ..., x_{2k}$ are arbitrary points in X.

From (2.2.4.1), we define

$y_{2k+2n-1} = S(x_{2n-1}, x_{2n}, ..., x_{2k+2n-2}) = gx_{2k+2n-1}$

$y_{2k+2n} = T(x_{2n}, x_{2n+1}, ..., x_{2k+2n-1}) = fx_{2k+2n}$ for $n = 1, 2, ...$

Let $\alpha_{2n} = M(fx_{2n}, gx_{2n+1}, qt)$ and $\alpha_{2n-1} = M(gx_{2n-1}, fx_{2n}, qt)$ for $n = 1, 2, ...$

Put $\theta = \frac{1}{q}$ and $\mu = min\{\theta\frac{1+\sqrt{\alpha_1}}{1-\sqrt{\alpha_1}}, \theta^2\frac{1+\sqrt{\alpha_2}}{1-\sqrt{\alpha_2}}, ..., \theta^{2k}\frac{1+\sqrt{\alpha_{2k}}}{1-\sqrt{\alpha_{2k}}}\}$.Then $\theta > 1$.

By the selection of μ, we have

$$\alpha_n \geq \left(\frac{\mu - \theta^n}{\mu + \theta^n}\right)^2 for \quad n = 1, 2, ..., 2k \qquad (1)$$

Consider

$$\begin{aligned}
&\alpha_{2k+1}\\
&= M(gx_{2k+1}, fx_{2k+2}, qt)\\
&= M(S(x_1, x_2, ...x_{2k-1}, x_{2k}), T(x_2, x_3, ...x_{2k}, x_{2k+1}), qt)\\
&\geq \phi(M(gx_1, fx_2, t), M(fx_2, gx_3, t), ..., M(fx_{2k}, gx_{2k+1}, t)\\
&\geq \phi(\alpha_1, \alpha_2, ...\alpha_{2k-1}, \alpha_{2k}), \quad since \quad M(x, y, .) \quad and \quad \phi \quad are\ increasing\\
&\geq \phi\left(\left(\frac{\mu-\theta}{\mu+\theta}\right)^2, \left(\frac{\mu-\theta^2}{\mu+\theta^2}\right)^2, ..., \left(\frac{\mu-\theta^{2k}}{\mu+\theta^{2k}}\right)^2\right) \quad from\ (1)
\end{aligned}$$

$$\geq \phi\left(\left(\frac{\mu-\theta^{2k}}{\mu+\theta^{2k}}\right)^2, \left(\frac{\mu-\theta^{2k}}{\mu+\theta^{2k}}\right)^2, ..., \left(\frac{\mu-\theta^{2k}}{\mu+\theta^{2k}}\right)^2\right)$$
$$\geq \left(\frac{\mu-\theta^{2k}}{\mu+\theta^{2k}}\right)^2, since\ \phi(t,t,...,t) \geq t$$
$$\geq \left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2.$$

Thus

$$\alpha_{2k+1} \geq \left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2 \tag{2}$$

Also

$$\alpha_{2k+2}$$
$$= M(fx_{2k+2}, gx_{2k+3}, qt)$$
$$= M(S(x_3, x_4, ...x_{2k+1}, x_{2k+2}), T(x_2, x_3, ...x_{2k}, x_{2k+1}), qt)$$
$$\geq \phi(M(gx_3, fx_2, t), M(fx_4, gx_3, t), ..., M(fx_{2k+2}, gx_{2k+1}, t)$$
$$\geq \phi(\alpha_2, \alpha_3, ...\alpha_{2k}, \alpha_{2k+1})$$
$$\geq \phi\left(\left(\frac{\mu-\theta^2}{\mu+\theta^2}\right)^2, \left(\frac{\mu-\theta^3}{\mu+\theta^3}\right)^2, ..., \left(\frac{\mu-\theta^{2k}}{\mu+\theta^{2k}}\right)^2, \left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2\right)$$
$$\geq \phi\left(\left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2, \left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2, ..., \left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2\right)$$
$$\geq \left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2$$
$$\geq \left(\frac{\mu-\theta^{2k+2}}{\mu+\theta^{2k+2}}\right)^2.$$

Thus

$$\alpha_{2k+2} \geq \left(\frac{\mu-\theta^{2k+2}}{\mu+\theta^{2k+2}}\right)^2 \tag{3}$$

Continuing in this way,we have

$$\alpha_n \geq \left(\frac{\mu-\theta^n}{\mu+\theta^n}\right)^2, n = 1, 2, 3... \tag{4}$$

Now consider

$$
\begin{aligned}
& M(y_{2k+2n-1}, y_{2k+2n}, t) \\
& \geq M(y_{2k+2n-1}, y_{2k+2n}, qt), \;\; \textit{since } q \; < \; 1 \textit{ and } M(x,y,.) \textit{ is increasing} \\
& = M\begin{pmatrix} S(x_{2n-1}, x_{2n}, x_{2n+1}, ..., x_{2k+2n-3}, x_{2k+2n-2}), \\ T(x_{2n}, x_{2n+1}, ..., x_{2k+2n-2}, x_{2k+2n-1}), qt \end{pmatrix} \\
& \geq \phi \begin{pmatrix} M(gx_{2n-1}, fx_{2n}, t), M(fx_{2n}, gx_{2n+1}, t), \\ M(gx_{2n+1}, fx_{2n+2}, t), M(fx_{2n+2}, gx_{2n+3}, t), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ M(gx_{2k+2n-3}, fx_{2k+2n-2}, t), M(fx_{2k+2n-2}, gx_{2k+2n-1}, t) \end{pmatrix} \\
& \geq \phi(\alpha_{2n-1}, \alpha_{2n}, \alpha_{2n+1}, ..., \alpha_{2k+2n-3}, \alpha_{2k+2n-2}), \;\; \textit{since } \phi \textit{ and } M \textit{ are increasing} \\
& \geq \phi\left(\left(\tfrac{\mu-\theta^{2n-1}}{\mu+\theta^{2n-1}}\right)^2, \left(\tfrac{\mu-\theta^{2n}}{\mu+\theta^{2n}}\right)^2, ..., \left(\tfrac{\mu-\theta^{2k+2n-2}}{\mu+\theta^{2k+2n-2}}\right)^2\right) \textit{from}(4) \\
& \geq \phi\left(\left(\tfrac{\mu-\theta^{2k+2n-2}}{\mu+\theta^{2k+2n-2}}\right)^2, \left(\tfrac{\mu-\theta^{2k+2n-2}}{\mu+\theta^{2k+2n-2}}\right)^2, ..., \left(\tfrac{\mu-\theta^{2k+2n-2}}{\mu+\theta^{2k+2n-2}}\right)^2\right) \\
& \geq \left(\tfrac{\mu-\theta^{2k+2n-2}}{\mu+\theta^{2k+2n-2}}\right)^2 \\
& \geq \left(\tfrac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2
\end{aligned}
$$

Thus

$$M(y_{2k+2n-1}, y_{2k+2n}, t) \geq \left(\frac{\mu - \theta^{2k+2n-1}}{\mu + \theta^{2k+2n-1}}\right)^2 \qquad (5)$$

Also

$$
\begin{aligned}
& M\left(y_{2k+2n}, y_{2k+2n+1}, t\right) \\
& \geq M\left(y_{2k+2n}, y_{2k+2n+1}, qt\right), \;\; \textit{since } q \; < \; 1 \textit{ and } \phi \textit{ is increasing} \\
& = M\begin{pmatrix} S\left(x_{2n+1}, x_{2n+2}, x_{2n+3}, ..., x_{2k+2n-1}, x_{2k+2n}\right), \\ T\left(x_{2n}, x_{2n+1}, x_{2n+2}, ..., x_{2k+2n-2}, x_{2k+2n-1}\right), qt \end{pmatrix}
\end{aligned}
$$

$$\geq \phi\begin{pmatrix} M(gx_{2n+1}, fx_{2n}, t), M(fx_{2n+2}, gx_{2n+1}, t), \\ M(gx_{2n+3}, fx_{2n+2}, t), M(fx_{2n+4}, gx_{2n+3}, t), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ M(gx_{2k+2n-1}, fx_{2k+2n-2}, t), M(fx_{2k+2n}, gx_{2k+2n-1}, t) \end{pmatrix}$$

$$\geq \phi\left(\alpha_{2n}, \alpha_{2n+1}, ..., \alpha_{2k+2n-2}, \alpha_{2k+2n-1}\right)$$

$$\geq \phi\left(\left(\tfrac{\mu-\theta^{2n}}{\mu+\theta^{2n}}\right)^2, \left(\tfrac{\mu-\theta^{2n+1}}{\mu+\theta^{2n+1}}\right)^2, ..., \left(\tfrac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2\right) \ from\ (4)$$

$$\geq \phi\left(\left(\tfrac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2, \left(\tfrac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2, ..., \left(\tfrac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2\right)$$

$$\geq \left(\tfrac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2, \ since\ \phi(t, t, t, ..., t) \ \geq \ t$$

$$\geq \left(\tfrac{\mu-\theta^{2k+2n}}{\mu+\theta^{2k+2n}}\right)^2.$$

Thus

$$M\left(y_{2k+2n}, y_{2k+2n+1}, t\right) \geq \left(\frac{\mu-\theta^{2k+2n}}{\mu+\theta^{2k+2n}}\right)^2 \tag{6}$$

Hence from (5) and (6) we have

$$M\left(y_{2k+n}, y_{2k+n+1}, t\right) \geq \left(\frac{\mu-\theta^{2k+n}}{\mu+\theta^{2k+n}}\right)^2 \quad for \quad n = 1, 2, ... \tag{7}$$

Now for $n, p \in N$, we have

$$M(y_{2k+n}, y_{2k+n+p}, t)$$

$$\geq M(y_{2k+n}, y_{2k+n+1}, \tfrac{t}{p}) * M(y_{2k+n+1}, y_{2k+n+2}, \tfrac{t}{p}) * ... * M\left(y_{2k+n+p-1}, y_{2k+n+p}, \tfrac{t}{p}\right)$$

$$\geq \left(\tfrac{\mu-\theta^{2k+n}}{\mu+\theta^{2k+n}}\right)^2 * \left(\tfrac{\mu-\theta^{2k+n+1}}{\mu+\theta^{2k+n+1}}\right)^2 * .... * \left(\tfrac{\mu-\theta^{2k+n+p-1}}{\mu+\theta^{2k+n+p-1}}\right)^2, from (7)$$

$$\to 1 * 1 * 1 * .... * 1 \ as\ n \ \to \infty$$

$$= 1.$$

Hence $\{y_{2k+n}\}$ is a Cauchy sequence in X.

Since X is complete, there exists $z \in X$ such that $y_{2k+n} \to z$ as $n \to \infty$.

From (2.2.4.4), there exists $u \in X$ such that

$$z = fu = gu \tag{8}$$

Now consider

$$\begin{aligned}
& M\left(S\left(u,u,...,u,u\right),y_{2k+2n},qt\right) \\
& = M\left(S\left(u,u,...,u,u\right),T\left(x_{2n},x_{2n+1},...,x_{2n+2k-2},x_{2n+2k-1}\right),qt\right) \\
& \geq \phi\left(\begin{array}{l} M\left(gu,fx_{2n},t\right),M\left(fu,gx_{2n+1},t\right), \\ ...., \\ M\left(gu,fx_{2n+2k-2},t\right),M\left(fu,gx_{2k+2n-1},t\right) \end{array}\right).
\end{aligned}$$

Letting $n \to \infty$ and using (8), we get

$$M(S(u,u,,...,u,u),fu,qt) \geq \phi(1,1,...,1,1) \geq 1$$

which implies that

$$S(u,u,...,u,u) = fu \tag{9}$$

Similarly we can prove that

$$T(u,u,...,u,u) = gu \tag{10}$$

Since (f,S) and (g,T) are jointly $2k$-weakly compatible pairs, we have

$$fz = f(fu) = f(S(u,u,...,u)) = S(fu,fu,...,fu) = S(z,z,...,z) \tag{11}$$

and also

$$gz = T(z,z,...,z,z) \tag{12}$$

Now consider

$$\begin{aligned}
& M(fz,z,qt) \\
& = M(S(z,z,...,z,z),T(u,u,...,u,u),qt),\ \ from\ (11),\ (8),\ (10) \\
& \geq \phi\left(\begin{array}{l} M(gz,fu,t),M(fz,gu,t), \\ M(gz,fu,t),M(fz,gu,t), \\ .... \\ M(gz,fu,t),M(fz,gu,t) \end{array}\right)
\end{aligned}$$

$$\geq \phi \begin{pmatrix} \min\{M(gz,z,t), M(fz,z,t)\}, \\ \min\{M(gz,z,t), M(fz,z,t)\}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \min\{M(gz,z,t), M(fz,z,t)\} \end{pmatrix}$$

$$\geq \min\{M(gz,z,t), M(fz,z,t)\}.$$

Thus

$$M(fz,z,qt) \geq \min\{M(gz,z,t), M(fz,z,t)\} \tag{13}$$

Similarly, we can show that

$$M(gz,z,qt) \geq \min\{M(z,fz,t), M(z,gz,t)\} \tag{14}$$

Thus from (13) and (14), we have

$$\min\{M(fz,z,qt), M(gz,z,qt)\} \geq \min\{M(z,fz,t), M(z,gz,t)\}$$

which in turn yields from condition(A) that

$$z = fz \ \ and \ \ z = gz \tag{15}$$

From (11), (12) and (15), we have

$$z = fz = gz = S(z,z,...,z) = T(z,z,...,z) \tag{16}$$

Suppose there exists $z' \in X$ such that

$$z' = fz' = gz' = S(z',z',...,z',z') = T(z',z',...,z',z').$$

Then from (2.2.4.2) we have

$$
\begin{aligned}
&M(z,z',qt)\\
&= M(S(z,z,...,z,z),T(z',z',...,z',z'),qt)\\
&\geq \phi\begin{pmatrix} M(gz,fz',t), M(fz,gz',t),\\ M(gz,fz',t), M(fz,gz',t),\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ M(gz,fz',t), M(fz,gz',t)\end{pmatrix}\\
&= \phi(M(z,z',t),M(z,z',t),...,M(z,z',t))\\
&\geq M(z,z',t)
\end{aligned}
$$

From the condition(A), we have $z' = z$.

Thus z is the unique point in X satisfying (16).

Now we give an example to illustrate our main Theorem 2.2.4.

Example 2.2.5. Let $X = [0,1], a * b = ab, M(x,y,t) = e^{-\frac{|x-y|}{t}}$ and $k = 1$. Define $\phi : [0,1]^2 \to [0,1]$ as $\phi(x_1,x_2) = \min\{x_1,x_2\}$. Let $S,T : X^2 \to X$ and $f,g : X \to X$ be defined as $S(x,y) = \frac{3x^2+2y}{72}$, $T(x,y) = \frac{2x+3y^2}{72}$, $fx = \frac{x}{6}$ and $gx = \frac{x^2}{4}$. Now for $x_1,x_2,y_1,y_2 \in X$, we have

$$
\begin{aligned}
|S(x_1,x_2) - T(y_1,y_2)| &= |\tfrac{3x_1^2+2x_2}{72} - \tfrac{2y_1+3y_2^2}{72}|\\
&= \tfrac{1}{72}\,|3x_1^2 - 2y_1 + 2x_2 - 3y_2^2|\\
&\leq \tfrac{1}{36}\max\{|3x_1^2 - 2y_1|, |2x_2 - 3y_2^2|\}.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
M(S(x_1,x_2),T(y_1,y_2),\tfrac{1}{3}t) &= e^{-\frac{|S(x_1,x_2)-T(y_1,y_2)|}{\frac{1}{3}t}}\\
&\geq e^{-\frac{1}{36}\frac{\max\{|3x_1^2-2y_1|,|2x_2-3y_2^2|\}}{\frac{t}{3}}}\\
&= e^{-\frac{\max\{|3x_1^2-2y_1|,|2x_2-3y_2^2|\}}{12t}}
\end{aligned}
$$

$$= e^{-\frac{\max\{|\frac{x_1^2}{4}-\frac{y_1}{6}|,|\frac{x_2}{6}-\frac{y_2^2}{4}|\}}{t}}$$

$$\geq \min\left\{ e^{-\frac{|\frac{x_1^2}{4}-\frac{y_1}{6}|}{t}}, \quad e^{-\frac{|\frac{x_2}{6}-\frac{y_2^2}{4}|}{t}} \right\}$$

$$= \min\{M(gx_1, fy_1, t), M(fx_2, gy_2, t)\}$$

$$= \phi(M(gx_1, fy_1, t), M(fx_2, gy_2, t).$$

Thus (2.2.4.2) is satisfied with $q = \frac{1}{3}$.

One can easily verify the remaining conditions of Theorem 2.2.4.

Clearly 0 is the unique point in X satisfying (16).

Corollary 2.2.6. Let $(X, M, *)$ be fuzzy metric space with the condition (A) and $S, T : X^{2k} \to X$ and $f : X \to X$ be mappings satisfying:

(2.2.6.1) $S(X^{2k}) \subseteq f(X), T(X^{2k}) \subseteq f(X)$,

(2.2.6.2) $M(S(x_1, x_2, ..., x_{2k}), T(y_1, y_2, ..., y_{2k}), qt)$

$\geq \phi\left(M(fx_1, fy_1, t), M(fx_2, fy_2, t), ..., M(fx_{2k}, fy_{2k}, t)\right)$

$\forall x_1, x_2, ..., x_{2k}, y_1, y_2, ..., y_{2k} \in X, \forall t > 0$ and $0 < q < 1$,

(2.2.6.3) $f(X)$ is a complete subspace of X.

(2.2.6.4) Either (f, S) or (f, T) is a $2k$-weakly compatible pair. Then there exists a unique $u \in X$ such that $u = fu = S(u, u, ..., u, u) = T(u, u, ..., u, u)$.

Corollary 2.2.7. Let $(X, M, *)$ be a complete fuzzy metric space with the condition(A) and $S, T : X^{2k} \to X$ be mappings satisfying:

(2.2.7.1) $M(S(x_1, x_2, ..., x_{2k}), T(y_1, y_2, ..., y_{2k}), qt)$

$\geq \phi\left(M\left(x_1, y_1, t\right), M\left(x_2, y_2, t\right), ..., M\left(x_{2k}, y_{2k}, t\right)\right)$

$\forall\ x_1, x_2, ..., x_{2k}, y_1, y_2, ..., y_{2k} \in X, \forall\ t > 0$ and $0 < q < 1$.

Then there exists a unique $u \in X$ such that $u = S(u, u, ..., u) = T(u, u, ..., u)$.

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SECTION 2.3: COINCIDENCE FIXED POINT THEOREMS FOR TWO PAIRS OF HYBRID MAPPINGS IN COMPLEX VALUED METRIC SPACES

In this section, we generalize the Theorem of Azam et al.[4] for two pairs of hybrid mappings using S-weakly commuting in complex valued metric spaces.

In 1969, Nadler [83] introduced the study of fixed points for multi-valued contraction mapping. Later many authors, for example [2, 5, 24, 38, 41, 52, 54, 56, 57, 58, 59, 60, 64, 86] proved fixed point results in different types of generalized metric spaces.

Recently Azam et al.[4] proved the following theorem.

Theorem 2.3.1.(Azam et al.[4]): Let (X, d) be a complete complex-valued metric space and let $S, T : X \to CB(X)$ be multi-valued mappings with g.l.b property such that

(2.3.1.1) $ad(x, Ty) + bd(y, Sx) + c\frac{d(x,Ty)d(y,Sx)}{1+d(x,y)} \in s(Sx, Ty)$

for all $x, y \in X$ and $a + b + c < 1$

Then S and T have a common fixed point.

In this section using f is S-weakly commuting we prove a coincidence point theorem for two pairs of hybrid mappings in complex valued metric spaces. Our theorem is generalization of Theorem 2.3.1 of Azam et al. [4].

In this section we need the following notations of Ahmad et al. [38].

Let (X, d) be a complex valued metric space. We denote

$s(z_1) = \{z_2 \in C : z_1 \precsim z_2\}$ for $z_1 \in C$ and

$s(a, B) = \bigcup\limits_{b \in B} s(d(a, b)) = \bigcup\limits_{b \in B} \{z \in C : d(a, b) \precsim z\}$ for $a \in X$ and $B \in C(X)$.

For $A, B \in C(X)$, we denote

$s(A,B) = \left(\bigcap_{a\in A} s(a,B)\right) \cap \left(\bigcap_{b\in B} s(b,A)\right)$.

Theorem 2.3.2. Let (X,d) be a complex valued metric space.

Let $S,T : X \to CB(X)$ be multi valued mappings $f,g : X \to X$ satisfying

(2.3.2.1) $Sx \subseteq g(X)$, $Tx \subseteq f(X)$,$\forall\ x \in X$,

(2.3.2.2) $ad(fx,Ty) + bd(gy,Sx) + \frac{cd(fx,Ty)d(gy,Sx)}{1+d(fx,gy)} \in s(Sx,Ty)$

for all $x,y \in X$ and $a,b,c \in \mathbb{R}^+$ such that $2a + 2b < 1$,

(2.3.2.3) f is S-weakly commuting and g is T-weakly commuting,

(2.3.2.4) $f(X)$ is complete.

Then (f,S) and (g,T) have the same coincidence point.

Proof: Let x_1 be an arbitrary point in X. Write $y_1 = fx_1$.

Since $Sx_1 \subseteq g(X)$, there exists $x_2 \in X$ such that $y_2 = gx_2 \in Sx_1$.

From (2.3.2.2), we have

$ad(fx_1,Tx_2) + bd(gx_2,Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)} \in s(Sx_1,Tx_2)$.

$ad(fx_1,Tx_2) + bd(gx_2,Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)} \in \left(\bigcap_{x\in Sx_1} s(x,Tx_2)\right)$.

$ad(fx_1,Tx_2) + bd(gx_2,Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)} \in s(x,Tx_2), \forall x \in Sx_1$.

$ad(fx_1,Tx_2) + bd(gx_2,Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)} \in s(gx_2,Tx_2)$.

$ad(fx_1,Tx_2) + bd(gx_2,Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)} \in \bigcup_{x\in Tx_2} s\left(d(gx_2,x)\right)$.

Since $Tx_2 \subseteq f(X)$, there exists some $x_3 \in X$ with $y_3 = fx_3 \in Tx_2$ such that

$ad(fx_1,Tx_2) + bd(gx_2,Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)}. \in s\left(d(gx_2,fx_3)\right)$.

Hence

$d(gx_2,fx_3) \precsim ad(fx_1,Tx_2) + bd(gx_2,Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)}$.

$d(y_2,y_3) \precsim ad(y_1,y_3) + bd(y_2,y_2) + \frac{cd(y_1,y_3)d(y_2,y_2)}{1+d(y_1,y_2)}$.

$|d(y_2, y_3)| \leq a\,|d(y_1, y_2)| + a\,|d(y_2, y_3)|.$

$|d(y_2, y_3)| \leq \frac{a}{1-a}\,|d(y_1, y_2)|.$ (1)

Now,

$ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)} \in s(Sx_3, Tx_2).$

$ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)} \in \left(\bigcap_{y \in Tx_2} s(Sx_3, y)\right).$

$ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)} \in s(Sx_3, y), \forall y \in Tx_2$

$ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)} \in s(Sx_3, fx_3).$

$ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)} \in \bigcup_{y \in Sx_3} s\,(d(y, fx_3)).$

Since $Sx_3 \subseteq g(X)$, there exists some $x_4 \in X$ with $y_4 = gx_4 \in Sx_3$ such that

$ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)} \in s\,(d(gx_4, fx_3)).$

Hence

$d(gx_4, fx_3) \precsim ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)}.$

$d(y_3, y_4) \precsim ad(y_3, y_3) + bd(y_2, y_4) + \frac{cd(y_3,y_3)d(y_2,y_4)}{1+d(y_3,y_2)}.$

$|d(y_3, y_4)| \leq b\,|d(y_2, y_3)| + b\,|d(y_3, y_4)|$

$\left|d(y_{3,}y_4)\right| \leq \frac{b}{1-b}\left|d(y_{2,}y_3)\right|.$ (2)

putting $h = max\left\{\frac{a}{1-a}, \frac{b}{1-b}\right\}$ and we continuing in this way, we get

$$\begin{aligned}\left|d(y_n, y_{n+1})\right| &\leq h\left|d(y_{n-1}, y_n)\right| \\ &\leq h^2\left|d(y_{n-2}, y_{n-1})\right| \\ &. \\ &. \\ &. \\ &\leq h^{n-1}\,|d(y_1, y_2)|\end{aligned}$$

Now for $m > n$ consider

$$\left|d(y_n,y_m)\right| \leq \left|d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + ... + d(y_{m-1},y_m)\right|$$
$$\leq h^{n-1} + h^n + ... + h^{m-2}\left|d(y_1,y_2)\right|$$
$$\leq \left[\tfrac{h^{n-1}}{1-h}\right] \to\ 0\ as\ m,n \to \infty.$$

Thus $\{y_n\}$ is a Cauchy sequence in X.

Since $f(X)$ is complete, $\{y_{2n+1}\} = \{fx_{2n+1}\}$ is Cauchy, it follows that $\{y_{2n+1}\}$ converges to $u \in f(X)$. Hence there exists $v \in X$ such that $u = fv$.

Since $\{y_n\}$ is a Cauchy sequence and $\{y_{2n+1}\} \to u$ it follow that $\{y_{2n}\} \to u$.

$$ad(fv,Tx_{2n}) + bd(gx_{2n},Sv)+\tfrac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})} \in s(Sv,Tx_{2n}).$$
$$ad(fv,Tx_{2n}) + bd(gx_{2n},Sv)+\tfrac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})} \in \left(\bigcap_{y\in Tx_{2n}} s(Sv,y)\right).$$
$$ad(fv,Tx_{2n}) + bd(gx_{2n},Sv)+\tfrac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})} \in s(Sv,y), \forall y \in Tx_{2n}.$$
$$ad(fv,Tx_{2n}) + bd(gx_{2n},Sv)+\tfrac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})} \in s(Sv,y_{2n+1}).$$
$$ad(fv,Tx_{2n}) + bd(gx_{2n},Sv)+\tfrac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})} \in \bigcup_{u^1\in Sv} s\left(d(u^1,y_{2n+1})\right).$$

There exists $v_n \in Sv$ such that

$$ad(fv,Tx_{2n}) + bd(gx_{2n},Sv)+\tfrac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})} \in s\left(d(v_n,y_{2n+1})\right).$$

Therefore $d(v_n,y_{2n+1}) \precsim ad(fv,Tx_{2n}) + bd(gx_{2n},Sv)+\tfrac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})}$.

Using g.l.b.property, we get

$$d(v_n,y_{2n+1}) \preceq ad(fv,y_{2n+1}) + bd(y_{2n},v_n)+\tfrac{cd(fv,y_{2n+1})d(y_{2n},v_n)}{1+d(fv,y_{2n})}.$$

Using triangular inequality, we obtain

$$d(v_n,y_{2n+1}) \precsim ad(fv,y_{2n+1}) + bd(y_{2n},y_{2n+1}) + bd(y_{2n+1},v_n)$$
$$+\ \tfrac{cd(fv,y_{2n+1})d(y_{2n},v_n)}{1+d(fv,y_{2n})}.$$

$$d(v_n,y_{2n+1}) \precsim \tfrac{a}{1-b}d(fv,y_{2n+1}) + \tfrac{b}{1-b}d(y_{2n},y_{2n+1}) + \tfrac{c}{1-b}\tfrac{d(fv,y_{2n+1})d(y_{2n},v_n)}{1+d(fv,y_{2n})}$$

Now consider

$$d(fv,v_n) \precsim d(fv,y_{2n+1}) + d(y_{2n+1},v_n)$$

$\precsim d(fv, y_{2n+1}) + \frac{a}{1-b} d(fv, y_{2n+1}) + \frac{b}{1-b} d(y_{2n}, y_{2n+1}) + \frac{c}{1-b} \frac{d(fv, y_{2n+1}) d(y_{2n}, v_n)}{1+d(fv, y_{2n})}$

$$|d(fv, v_n)| \leq |d(fv, y_{2n+1})| + \frac{a}{1-b} |d(fv, y_{2n+1})| + \frac{b}{1-b} |d(y_{2n}, y_{2n+1})| + \frac{c}{1-b} \frac{|d(fv, y_{2n+1})| \left|d(y_{2n}, v_n)\right|}{\left|1+d(fv, y_{2n})\right|}.$$

Letting $n \to \infty$, we obtain

$|d(fv, v_n)| \to 0$ as $n \to \infty$. By Lemma 1.5.2(Ch-1), we have $v_n \to fv$ as $n \to \infty$.

Since Sv is closed and $\{v_n\} \subseteq Sv$, it follows that $fv \in Sv$.

Now $u = fv \in Sv$ and $Sv \subseteq g(X)$ it follows that $u = fv = gw$ for some $w \in X$.

$$ad(fx_{2n-1}, Tw) + bd(gw, Sx_{2n-1}) + \frac{cd(fx_{2n-1}, Tw) d(gw, Sx_{2n-1})}{1+d(fx_{2n-1}, gw)} \in s(Sx_{2n-1}, Tw).$$

$$ad(fx_{2n-1}, Tw) + bd(gw, Sx_{2n-1}) + \frac{cd(fx_{2n-1}, Tw) d(gw, Sx_{2n-1})}{1+d(fx_{2n-1}, gw)}. \in \left(\bigcap_{y^1 \in Sx_{2n-1}} s(y^1, Tw) \right).$$

$$ad(fx_{2n-1}, Tw) + bd(gw, Sx_{2n-1}) + \frac{cd(fx_{2n-1}, Tw) d(gw, Sx_{2n-1})}{1+d(fx_{2n-1}, gw)}. \in s(y^1, Tw), \forall y^1 \in Sx_{2n-1}.$$

$$ad(fx_{2n-1}, Tw) + bd(gw, Sx_{2n-1}) + \frac{cd(fx_{2n-1}, Tw) d(gw, Sx_{2n-1})}{1+d(fx_{2n-1}, gw)} \in s(y_{2n}, Tw).$$

$$ad(fx_{2n-1}, Tw) + bd(gw, Sx_{2n-1}) + \frac{cd(fx_{2n-1}, Tw) d(gw, Sx_{2n-1})}{1+d(fx_{2n-1}, gw)} \in \bigcup_{u^1 \in Tw} s\left(d(y_{2n}, u^1)\right).$$

There exists some $w_n \in Tw$ such that

$$ad(fx_{2n-1}, Tw) + bd(gw, Sx_{2n-1}) + \frac{cd(fx_{2n-1}, Tw) d(gw, Sx_{2n-1})}{1+d(fx_{2n-1}, gw)} \in s(d(y_{2n}, w_n)).$$

$$d(y_{2n}, w_n) \precsim ad(fx_{2n-1}, Tw) + bd(gw, Sx_{2n-1}) + \frac{cd(fx_{2n-1}, Tw) d(gw, Sx_{2n-1})}{1+d(fx_{2n-1}, gw)}.$$

Using g.l.b.property, we obtain

$$d(y_{2n}, w_n) \precsim ad(y_{2n-1}, w_n) + bd(gw, y_{2n}) + \frac{cd(y_{2n-1}, w_n) d(gw, y_{2n})}{1+d(y_{2n-1}, gw)}.$$

Using triangular inequality, we have

$d(y_{2n}, w_n) \precsim ad(y_{2n-1}, y_{2n}) + ad(y_{2n}, w_n) + bd(gw, y_{2n}) + \frac{cd(y_{2n-1}, w_n)d(gw, y_{2n})}{1+d(y_{2n-1}, gw)}$.

$d(y_{2n}, w_n) \precsim \frac{a}{1-a}d(y_{2n-1}, y_{2n}) + \frac{b}{1-a}d(gw, y_{2n}) + \frac{c}{1-a}\frac{d(y_{2n-1}, w_n)d(gw, y_{2n})}{1+d(y_{2n-1}, gw)}$.

Now consider

$d(gw, w_n)$

$\precsim d(gw, y_{2n}) + d(y_{2n}, w_n)$.

$\precsim d(gw, y_{2n}) + \frac{a}{1-a}d(y_{2n-1}, y_{2n}) + \frac{b}{1-a}d(gw, y_{2n}) + \frac{c}{1-a}\frac{d(y_{2n-1}, w_n)d(gw, y_{2n})}{1+d(y_{2n-1}, gw)}$.

$|d(gw, w_n)| \leq |d(gw, y_{2n})| + \frac{a}{1-a}|d(y_{2n-1}, y_{2n})| + \frac{b}{1-a}|d(gw, y_{2n})|$

$+\frac{c}{1-a}\frac{|d(y_{2n-1}, w_n)||d(gw, y_{2n})|}{|1+d(y_{2n-1}, gw)|}$.

Letting $n \to \infty$ we get

$|d(gw, w_n)| \to 0$ as $n \to \infty$.

By Lemma 1.5.2(Ch-1), we have $w_n \to gw$ as $n \to \infty$.

Since Tw is closed and $\{w_n\} \subseteq Tw$, it follows that $gw \in Tw$.

We have $u = fv = gw \in Tw$.

Since f is S-weakly commuting and g is T-weakly commuting we have

$f^2v \in Sfv \Rightarrow fu \in Su$ and $g^2w \in Tgw \Rightarrow gu \in Tu$.

Thus the pairs (f, S) and (g, T) have the same coincidence point.

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CHAPTER 3

COUPLED AND COINCIDENCE POINT THEOREMS IN C^*-ALGEBRA VALUED FUZZY SOFT METRIC SPACES

We divide Chapter 3 into two sections, namely, Section 3.1 and Section 3.2. The main aim of the chapter is to prove common coupled fixed point and coincidence point theorems in C^* -algebra valued Fuzzy soft metric spaces.

SECTION 3.1: COUPLED FIXED POINT RESULTS AND APPLICATIONS IN C^* - ALGEBRA VALUED FUZZY SOFT METRIC SPACES

In this section, we establish the existence and uniqueness of common coupled fixed point results for three mappings in C^*- Algebra valued Fuzzy Soft metric spaces. Moreover, we give an illustration which presents the applicability of the achieved result and also we provide application to Integral Equations.

Definition 3.1.1.(R.P.Agarwal et al.[79]): *Let* $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space. Let $S : \tilde{E} \times \tilde{E} \to \tilde{E}$ be a mapping, an element $(F_{e_1}, G_{e_1}) \in \tilde{E} \times \tilde{E}$ is called coupled fixed point of S if $S(F_{e_1}, G_{e_1}) = F_{e_1}$ and $S(G_{e_1}, F_{e_1}) = G_{e_1}$.

Definition 3.1.2.(R.P.Agarwal et al.[79]): *Let* $\tilde{E}$ be absolute fuzzy soft set. An element
$(F_{e_1}, G_{e_1}) \in \tilde{E} \times \tilde{E}$ is called

(*i*) a coupled coincidence point of mappings $S : \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f : \tilde{E} \to \tilde{E}$ if $fF_{e_1} = S(F_{e_1}, G_{e_1})$ and $fG_{e_1} = S(G_{e_1}, F_{e_1})$,

(*ii*) a common coupled fixed point of mappings $S : \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f : \tilde{E} \to \tilde{E}$ if $F_{e_1} = fF_{e_1} = S(F_{e_1}, G_{e_1})$ and $G_{e_1} = fG_{e_1} = S(G_{e_1}, F_{e_1})$.

Definition 3.1.3.(R.P.Agarwal et al.[79]): Let $\tilde{E}$ be absolute fuzzy soft set and $S : \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f : \tilde{E} \to \tilde{E}$. Then $\{S, f\}$ is said to be ω-compatible pairs if

$f\,(S(F_{e_1}, G_{e_1})) = S(fF_{e_1}, fG_{e_1})$ and $f\,(S(G_{e_1}, F_{e_1})) = S(fG_{e_1}, fF_{e_1})$.

Theorem 3.1.4. Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space.Suppose $S\colon \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f, g\colon \tilde{E} \to \tilde{E}$ be satisfying,

(3.1.4.1) $S(\tilde{E} \times \tilde{E}) \subseteq g(\tilde{E})$ and $S(\tilde{E} \times \tilde{E}) \subseteq f(\tilde{E})$,

(3.1.4.2) $\{S, f\}$ and $\{S, g\}$ are ω-compatible pairs,

(3.1.4..3) one of $f(\tilde{E})$ or $g(\tilde{E})$ is complete C^*-algebra valued fuzzy soft metrics of $\tilde{E}$,

(3.1.4.4) $\tilde{d_{c^*}}\,(S(F_{e_1}, G_{e_1}), S(F_{e_2}, G_{e_2})) \preceq \tilde{a}^\star \tilde{d_{c^*}}(fF_{e_1}, gF_{e_2})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fG_{e_1}, gG_{e_2})\tilde{a}$ for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$,

where $\tilde{a} \in \tilde{C}$ with $||\sqrt{2}\tilde{a}|| < 1$. Then S, f and g have a unique common coupled fixed point in $\tilde{E} \times \tilde{E}$. Moreover, S, f and g have a unique common fixed point in $\tilde{E}$.

Proof: Let $F_{e_0}, G_{e_0} \in \tilde{E}$. From (3.1.4.1) we can construct the sequences $\{F_{e_{2n}}\}_{2n=1}{}^{\infty}$, $\{G_{e_{2n}}\}_{2n=1}{}^{\infty}$, $\{I_{e_{2n}}\}_{2n=1}{}^{\infty}$, $\{J_{e_{2n}}\}_{2n=1}{}^{\infty}$ such that

$$S(F_{e_{2n}}, G_{e_{2n}}) = fF_{e_{2n+1}} = I_{e_{2n}} \quad S(F_{e_{2n+1}}, G_{e_{2n+1}}) = gF_{e_{2n+2}} = I_{e_{2n+1}}$$
$$S(G_{e_{2n}}, F_{e_{2n}}) = fG_{e_{2n+1}} = J_{e_{2n}} \quad S(G_{e_{2n+1}}, F_{e_{2n+1}}) = gG_{e_{2n+2}} = J_{e_{2n+1}}$$

for $n = 0, 1, 2, \ldots$

Notices that in C^*-algebra, if $\tilde{a}, \tilde{b} \in \tilde{C_+}$ and $\tilde{a} \preceq \tilde{b}$, then for any $\tilde{x} \in \tilde{C_+}$ both $\tilde{x}^\star \tilde{a} \tilde{x}$ and $\tilde{x}^\star \tilde{b} \tilde{x}$ are positive elements and $\tilde{x}^\star \tilde{a} \tilde{x} \preceq \tilde{x}^\star \tilde{b} \tilde{x}$.

From (3.1.4.4), we get

$$\begin{aligned}\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n+2}}) &= \tilde{d_{c^*}}\left(S(F_{e_{2n+1}}, G_{e_{2n+1}}), S(F_{e_{2n+2}}, G_{e_{2n+2}})\right)\\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fF_{e_{2n+1}}, gF_{e_{2n+2}})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fG_{e_{2n+1}}, gG_{e_{2n+2}})\tilde{a}\\ &\preceq \tilde{a}^\star\left(\tilde{d_{c^*}}(I_{e_{2n}}, I_{e_{2n+1}}) + \tilde{d_{c^*}}(J_{e_{2n}}, J_{e_{2n+1}})\right)\tilde{a} \qquad (3.1.4.5)\end{aligned}$$

Similarly,

$$\begin{aligned}\tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n+2}}) &= \tilde{d_{c^*}}\left(S(G_{e_{2n+1}}, F_{e_{2n+1}}), S(G_{e_{2n+2}}, F_{e_{2n+2}})\right)\\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fG_{e_{2n+1}}, gG_{e_{2n+2}})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fF_{e_{2n+1}}, gF_{e_{2n+2}})\tilde{a}\\ &\preceq \tilde{a}^\star\left(\tilde{d_{c^*}}(J_{e_{2n}}, J_{e_{2n+1}}) + \tilde{d_{c^*}}(I_{e_{2n}}, I_{e_{2n+1}})\right)\tilde{a} \qquad (3.1.4.6)\end{aligned}$$

Let $\kappa_{2n+1} = \tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n+2}}) + \tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n+2}})$

and now from (3.1.4.5) and (3.1.4.6), we have

$$\begin{aligned}\kappa_{2n+1} &= \tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n+2}}) + \tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n+2}})\\ &\preceq \tilde{a}^\star\left(\tilde{d_{c^*}}(I_{e_{2n}}, I_{e_{2n+1}}) + \tilde{d_{c^*}}(J_{e_{2n}}, J_{e_{2n+1}})\right)\tilde{a}\\ &+\tilde{a}^\star\left(\tilde{d_{c^*}}(J_{e_{2n}}, J_{e_{2n+1}}) + \tilde{d_{c^*}}(I_{e_{2n}}, I_{e_{2n+1}})\right)\tilde{a}\\ &\preceq (\sqrt{2}\tilde{a})^\star \kappa_{2n}(\sqrt{2}\tilde{a})\\ &\vdots\\ &\preceq \left[(\sqrt{2}\tilde{a})^\star\right]^{2n+1}\kappa_0(\sqrt{2}\tilde{a})^{2n+1}\end{aligned}$$

Now, we can obtain for any $n \in N$

$$\begin{aligned}\kappa_n &= \tilde{d_{c^*}}(I_{e_n}, I_{e_{n+1}}) + \tilde{d_{c^*}}(J_{e_n}, J_{e_{n+1}})\\ &\preceq (\sqrt{2}\tilde{a})^\star \kappa_{n-1}(\sqrt{2}\tilde{a})\\ &\vdots\\ &\preceq \left[(\sqrt{2}\tilde{a})^\star\right]^{n}\kappa_0(\sqrt{2}\tilde{a})^{n}\end{aligned}$$

If $\kappa_0 = \tilde{0}_{\tilde{C}}$, then from Definition 1.6.1(Ch-1) of M_1 we know (I_{e_0}, J_{e_0}) is a coupled fixed point of S, f and g.

Now letting $\tilde{0}_{\tilde{C}} \preceq \kappa_0$, we get for any $n \in N$, for any $p \in N$ and using triangle inequality

$$\tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n}}) \preceq \tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n+p-1}}) + \tilde{d_{c^*}}(I_{e_{2n+p-1}}, I_{e_{2n+p-2}}) + \cdots + \tilde{d_{c^*}}(I_{e_{2n+1}}, {}_{e_{2n}}),$$

$$\tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n}}) \preceq \tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n+p-1}}) + \tilde{d_{c^*}}(J_{e_{2n+p-1}}, J_{e_{2n+p-2}}) + \cdots + \tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n}}).$$

Consequently,

$$\begin{aligned} \tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n}}) + \tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n}}) &\preceq \kappa_{2n+p-1} + \kappa_{2n+p-2} + \cdots + \kappa_{2n} \\ &\preceq \sum_{m=2n}^{2n+p-1} \left[(\sqrt{2}\tilde{a})^\star\right]^m \kappa_0 (\sqrt{2}\tilde{a})^m \end{aligned}$$

and then

$$\begin{aligned} ||\tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n}}) + \tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n}})|| &\leq \kappa_{2n+p-1} + \kappa_{2n+p-2} + \cdots + \kappa_{2n} \\ &\leq \sum_{m=2n}^{2n+p-1} ||\sqrt{2}\tilde{a}||^{2m} \kappa_0 \leq \sum_{m=n}^{\infty} ||\sqrt{2}\tilde{a}||^{2m} \kappa_0 \\ &= \tfrac{||\sqrt{2}\tilde{a}||^{2n}}{1-||\sqrt{2}\tilde{a}||^2} \kappa_0 \to 0 \text{ as } n \to \infty \end{aligned}$$

which together with $\tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n}}) \preceq \tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n}}) + \tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n}})$ and $\tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n}}) \preceq \tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n}}) + \tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n}})$ implies $\{I_{e_{2n}}\}$ and $\{J_{e_{2n}}\}$ are Cauchy sequences in $\tilde{E}$ with respect to $\tilde{C}$.

It follows that $\{I_{e_{2n+1}}\}$ and $\{J_{e_{2n+1}}\}$ are also Cauchy sequences in $\tilde{E}$ with respect to $\tilde{C}$.

Thus $\{I_{e_n}\}$ and $\{J_{e_n}\}$ are Cauchy sequences in $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$.

Suppose $g(\tilde{E})$ is complete subspace of $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$.

Then the sequences $\{I_{e_n}\}$ and $\{J_{e_n}\}$ are converge to $I_{e'}, J_{e'}$ respectively in $g(\tilde{E})$.

Thus there exist $F_{e'}, G_{e'}$ in $g(\tilde{E})$ Such that

$$\lim_{n\to\infty} I_{e_n} = I_{e'} = gF_{e'} \text{ and } \lim_{n\to\infty} J_{e_n} = J_{e'} = gG_{e'} \quad (3.1.4.7)$$

We now claim that $S(F_{e'}, G_{e'}) = I_{e'}$ and $S(G_{e'}, F_{e'}) = J_{e'}$.

From (3.1.4.4) and using the triangular inequality, we have

$$\begin{aligned}
\tilde{0}_{\tilde{C}} &\preceq \tilde{d_{c^*}}(I_{e'}, S(F_{e'}, G_{e'})) \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{d_{c^*}}(I_{e_{2n+1}}, S(F_{e'}, G_{e'})) \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{d_{c^*}}(S(F_{e_{2n+1}}, G_{e_{2n+1}}), S(F_{e'}, G_{e'})) \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{a}^\star \tilde{d_{c^*}}(fF_{e_{2n+1}}, gF_{e'})\tilde{a} + \tilde{a}^\star \quad \tilde{d_{c^*}}(fG_{e_{2n+1}}, gG_{e'})\tilde{a} \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{a}^\star \tilde{d_{c^*}}(I_{e_{2n}}, I_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(J_{e_{2n}}, J_{e'})\tilde{a}.
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above relation, we obtain

$\tilde{d_{c^*}}(I_{e'}, S(F_{e'}, G_{e'})) = \tilde{0}_{\tilde{C}}$ and hence $S(F_{e'}, G_{e'}) = I_{e'}$.

Similarly, we prove $S(G_{e'}, F_{e'}) = J_{e'}$.

Therefore, it follows $S(F_{e'}, G_{e'}) = I_{e'} = gI_{e'}$ and $S(G_{e'}, F_{e'}) = J_{e'} = gJ_{e'}$.

Since $\{S, g\}$ is ω-compatible pair,

we have $S(I_{e'}, J_{e'}) = gI_{e'}$ and $S(J_{e'}, I_{e'}) = gJ_{e'}$.

Now to prove that $gI_{e'} = I_{e'}$ and $gJ_{e'} = J_{e'}$.

$$\begin{aligned}
\tilde{0}_{\tilde{C}} &\preceq \tilde{d}_{c^*}\left(I_{e_{2n+1}}, gI_{e'}\right) \\
&\preceq \tilde{d}_{c^*}\left(S\left(F_{e_{2n+1}}, G_{e_{2n+1}}\right), S\left(I_{e'}, J_{e'}\right)\right) \\
&\preceq \tilde{a}^\star \tilde{d}_{c^*}\left(fF_{e_{2n+1}}, gI_{e'}\right)\tilde{a} + \tilde{a}^\star \tilde{d}_{c^*}\left(fG_{e_{2n+1}}, gJ_{e'}\right)\tilde{a} \\
&\preceq \tilde{a}^\star \tilde{d}_{c^*}\left(I_{e_{2n}}, gI_{e'}\right)\tilde{a} + \tilde{a}^\star \tilde{d}_{c^*}\left(J_{e_{2n}}, gJ_{e'}\right)\tilde{a}.
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above relation, we obtain

$$\tilde{d}_{c^*}\left(I_{e'}, gI_{e'}\right) \preceq \tilde{a}^\star \tilde{d}_{c^*}\left(I_{e'}, gI_{e'}\right)\tilde{a} + \tilde{a}^\star \tilde{d}_{c^*}\left(J_{e'}, gJ_{e'}\right)\tilde{a}. \tag{3.1.4.8}$$

Similarly, we have

$$\tilde{d}_{c^*}\left(J_{e'}, gJ_{e'}\right) \preceq \tilde{a}^\star \tilde{d}_{c^*}\left(J_{e'}, gJ_{e'}\right)\tilde{a} + \tilde{a}^\star \tilde{d}_{c^*}\left(I_{e'}, gI_{e'}\right)\tilde{a}. \tag{3.1.4.9}$$

From (3.1.4.8) and (3.1.4.9), we have

$$\begin{aligned}\tilde{0}_{\tilde{C}} &\preceq \tilde{d_{c^*}}(I_{e'}, gI_{e'}) + \tilde{d_{c^*}}(J_{e'}, gJ_{e'}) \\ &\preceq (\sqrt{2}\tilde{a}^{\star})\left(\tilde{d_{c^*}}(I_{e'}, gI_{e'}) + \tilde{d_{c^*}}(J_{e'}, gJ_{e'})\right)(\sqrt{2}\tilde{a}).\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{0} &\leq ||\tilde{d_{c^*}}(I_{e'}, gI_{e'}) + \tilde{d_{c^*}}(J_{e'}, gJ_{e'})|| \\ &\leq ||(\sqrt{2}\tilde{a}^{\star})\left(\tilde{d_{c^*}}(I_{e'}, gI_{e'}) + \tilde{d_{c^*}}(J_{e'}, gJ_{e'})\right)(\sqrt{2}\tilde{a})|| \\ &\leq ||(\sqrt{2}\tilde{a})||^2||\tilde{d_{c^*}}(I_{e'}, gI_{e'}) + \tilde{d_{c^*}}(J_{e'}, gJ_{e'})||.\end{aligned}$$

Since $||(\sqrt{2}\tilde{a})|| < 1$, from above we have $||\tilde{d_{c^*}}(I_{e'}, gI_{e'}) + \tilde{d_{c^*}}(J_{e'}, gJ_{e'})|| = 0$.

Hence $gI_{e'} = I_{e'}$ and $gJ_{e'} = J_{e'}$.

Therefore, $S(I_{e'}, J_{e'}) = gI_{e'} = I_{e'}$ and $S(J_{e'}, I_{e'}) = gJ_{e'} = J_{e'}$.

Thus $(I_{e'}, J_{e'})$ is common coupled fixed point of S and g.

Since $S(\tilde{E} \times \tilde{E}) \subseteq f(\tilde{E})$. So there exist $K_{e'}$, $L_{e'} \in \tilde{E}$ such that

$S(I_{e'}, J_{e'}) = I_{e'} = fK_{e'}$ and $S(J_{e'}, I_{e'}) = J_{e'} = fL_{e'}$.

Now from (3.1.4.4), we have

$$\begin{aligned}\tilde{0}_{\tilde{C}} &\preceq \tilde{d_{c^*}}(S(K_{e'}, L_{e'}), I_{e'}) \\ &\preceq \tilde{d_{c^*}}(S(K_{e'}, L_{e'}), S((I_{e'}, J_{e'})) \\ &\preceq \tilde{a}^{\star}\tilde{d_{c^*}}(fK_{e'}, gI_{e'})\tilde{a} + \tilde{a}^{\star}\tilde{d_{c^*}}(fL_{e'}, gJ_{e'})\tilde{a} \\ &\preceq \tilde{a}^{\star}\tilde{d_{c^*}}(I_{e'}, I_{e'})\tilde{a} + \tilde{a}^{\star}\tilde{d_{c^*}}(J_{e'}, J_{e'})\tilde{a}.\end{aligned}$$

We have $\tilde{d_{c^*}}(S(K_{e'}, L_{e'}), I_{e'}) = 0$, which means that $I_{e'} = S(K_{e'}, L_{e'})$.

Similarly we can prove that $S(L_{e'}, K_{e'}) = J_{e'}$.

Since $\{S, f\}$ is ω-compatible pair, we have $S(I_{e'}, J_{e'}) = fI_{e'}$ and

$S(J_{e'}, I_{e'}) = fJ_{e'}$.

Now we prove that $fI_{e'} = I_{e'}$ and $fJ_{e'} = J_{e'}$.

$$\begin{aligned} \tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(fI_{e'}, I_{e'}) &\preceq \tilde{d_{c^*}}(S((I_{e'}, J_{e'})), S(I_{e'}, J_{e'})) \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, gI_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, gJ_{e'})\tilde{a} \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, I_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, J_{e'})\tilde{a} \qquad (3.1.4.10) \end{aligned}$$

and

$$\begin{aligned} \tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(fJ_{e'}, J_{e'}) &\preceq \tilde{d_{c^*}}(S((J_{e'}, I_{e'})), S(J_{e'}, I_{e'})) \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, gJ_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, gI_{e'})\tilde{a} \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, J_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, I_{e'})\tilde{a}. \qquad (3.1.4.11) \end{aligned}$$

From (3.1.4.10) and (3.1.4.11), we have

$$\begin{aligned} \tilde{0}_{\tilde{C}} &\preceq \tilde{d_{c^*}}(fI_{e'}, I_{e'}) + \tilde{d_{c^*}}(fJ_{e'}, J_{e'}) \\ &\preceq (\sqrt{2}\tilde{a}^\star)\left(\tilde{d_{c^*}}(fI_{e'}, I_{e'}) + \tilde{d_{c^*}}(fJ_{e'}, J_{e'})\right)(\sqrt{2}\tilde{a}). \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{0} &\leq ||\tilde{d_{c^*}}(fI_{e'}, I_{e'}) + \tilde{d_{c^*}}(fJ_{e'}, J_{e'})|| \\ &\leq ||(\sqrt{2}\tilde{a}^\star)\left(\tilde{d_{c^*}}(fI_{e'}, I_{e'}) + \tilde{d_{c^*}}(fJ_{e'}, J_{e'})\right)(\sqrt{2}\tilde{a})|| \\ &\leq ||(\sqrt{2}\tilde{a})||^2 ||\tilde{d_{c^*}}(fI_{e'}, I_{e'}) + \tilde{d_{c^*}}(fJ_{e'}, J_{e'})||. \end{aligned}$$

Since $||(\sqrt{2}\tilde{a})|| < 1$, from above we have $||\tilde{d_{c^*}}(fI_{e'}, I_{e'}) + \tilde{d_{c^*}}(fJ_{e'}, J_{e'})|| = 0$. Hence $fI_{e'} = I_{e'}$ and $fJ_{e'} = J_{e'}$

Therefore, we have $S(I_{e'}, J_{e'}) = fI_{e'} = I_{e'}$ and $S(J_{e'}, I_{e'}) = fJ_{e'} = J_{e'}$. Thus $(I_{e'}, J_{e'})$ is common coupled fixed point of S, f and g.

In the following we show that the uniqueness of common coupled fixed point in $\tilde{E} \times \tilde{E}$. First we assume that there is another coupled fixed point $(I_{e''}, J_{e''})$ of S, f and g in $\tilde{E} \times \tilde{E}$.

From (3.1.4.4), we have

$$\begin{aligned}\tilde{d_{c^*}}(I_{e'}, I_{e''}) &\preceq \tilde{d_{c^*}}(S(I_{e'}, J_{e'}), S(I_{e''}, J_{e''}))\\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, gI_{e''})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(gJ_{e'}, gJ_{e''})\tilde{a}\\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(I_{e'}, I_{e''})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(J_{e'}, J_{e''})\tilde{a} \qquad (3.1.4.12)\end{aligned}$$

and

$$\begin{aligned}\tilde{d_{c^*}}(J_{e'}, J_{e''}) &\preceq \tilde{d_{c^*}}(S(J_{e'}, I_{e'}), S(J_{e''}, I_{e''}))\\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, gJ_{e''})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(gI_{e'}, gI_{e''})\tilde{a}\\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(J_{e'}, J_{e''})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(I_{e'}, I_{e''})\tilde{a}. \qquad (3.1.4.13)\end{aligned}$$

From (3.1.4.12) and (3.1.4.13), we have

$$\tilde{d_{c^*}}(I_{e'}, I_{e''}) + \tilde{d_{c^*}}(J_{e'}, J_{e''}) \preceq (\sqrt{2}\tilde{a})^\star \left(\tilde{d_{c^*}}(I_{e'}, I_{e''}) + \tilde{d_{c^*}}(J_{e'}, J_{e''})\right)(\sqrt{2}\tilde{a}).$$

It follows that

$$||\tilde{d_{c^*}}(I_{e'}, I_{e''}) + \tilde{d_{c^*}}(J_{e'}, J_{e''})|| \leq ||\sqrt{2}\tilde{a}||^2 ||\tilde{d_{c^*}}(I_{e'}, I_{e''}) + \tilde{d_{c^*}}(J_{e'}, J_{e''})||.$$

Since $||\sqrt{2}\tilde{a}|| < 1$, from above we have $||\tilde{d_{c^*}}(I_{e'}, I_{e''}) + \tilde{d_{c^*}}(J_{e'}, J_{e''})|| = 0$.

Hence we get $(I_{e'}, J_{e'}) = (I_{e''}, J_{e''})$ which means that the coupled fixed point is unique.

In order to prove that S, f and g have a unique fixed point, we only have to prove $I_{e'} = J_{e'}$. We have

$$\begin{aligned}\tilde{d_{c^*}}(I_{e'}, J_{e'}) = \tilde{d_{c^*}}(S(I_{e'}, J_{e'}), S(J_{e'}, I_{e'})) &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, gJ_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, gI_{e'})\tilde{a}\\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(I_{e'}, J_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(J_{e'}, I_{e'})\tilde{a}\end{aligned}$$

then

$$\begin{aligned}||\tilde{d_{c^*}}(I_{e'}, J_{e'})|| &\leq ||\tilde{a}||^2 ||\tilde{d_{c^*}}(I_{e'}, J_{e'})|| + ||\tilde{a}||^2 ||\tilde{d_{c^*}}(J_{e'}, I_{e'})||\\ &\leq 2||\tilde{a}||^2 ||\tilde{d_{c^*}}(I_{e'}, J_{e'})||.\end{aligned}$$

It follows from the fact $||\tilde{a}|| < \frac{1}{\sqrt{2}}$ that $||\tilde{d_{c^*}}(I_{e'}, J_{e'})|| = 0$, thus $I_{e'} = J_{e'}$. which means that S, f and g have a unique fixed point in $\tilde{E}$.

Corollary 3.1.5. Let $(\tilde{E}, \tilde{A}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space. Suppose $S\colon \tilde{E} \times \tilde{E} \to \tilde{E}$ satisfies

(3.1.5.1) $$\tilde{d_{c^*}}\,(S(F_{e_1}, G_{e_1}), S(F_{e_2}, G_{e_2})) \preceq \tilde{a}^\star \tilde{d_{c^*}}(F_{e_1}, F_{e_2})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(G_{e_1}, G_{e_2})\tilde{a}$$

for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$, where $\tilde{a} \in \tilde{C}$ with $||\sqrt{2}\tilde{a}|| < 1$.

Then S has a unique fixed point in $\tilde{E}$.

Example 3.1.6. Let $E = \{e_1, e_2, e_3\}, U = \{a, b, c, d\}$ and C and D are two subset of E

where $C = \{e_1, e_2, e_3\}, D = \{e_1, e_2\}$. Define fuzzy soft set as,

$$(F_E, C) = \left\{ \begin{array}{c} e_1 = \{a_{0.3}, b_{0.2}, c_{0.4}, d_{0.1}\}, e_2 = \{a_{0.5}, b_{0.4}, c_{0.6}, d_{0.3}\}, \\ e_3 = \{a_{0.7}, b_{0.8}, c_{0.9}, d_{0.5}\} \end{array} \right\}$$

$$(G_E, D) = \{e_1 = \{a_{0.4}, b_{0.6}, c_{0.3}, d_{0.2}\}, e_2 = \{a_{0.8}, b_{0.9}, c_{0.5}, d_{0.7}\}\}$$

$$F_{e_1} = \mu_{F_{e_1}} = \{a_{0.3}, b_{0.2}, c_{0.4}, d_{0.1}\}, F_{e_2} = \mu_{F_{e_2}} = \{a_{0.5}, b_{0.4}, c_{0.6}, d_{0.3}\}$$

$$F_{e_3} = \mu_{F_{e_3}} = \{a_{0.7}, b_{0.8}, c_{0.9}, d_{0.5}\}$$

$$G_{e_1} = \mu_{G_{e_1}} = \{a_{0.4}, b_{0.6}, c_{0.3}, d_{0.2}\}, G_{e_2} = \mu_{G_{e_2}} = \{a_{0.8}, b_{0.9}, c_{0.5}, d_{0.7}\}$$

and $FSC(F_E) = \{F_{e_1}, F_{e_2}, F_{e_3}, G_{e_1}, G_{e_2}\}$,

let $\tilde{E}$ be absolute fuzzy soft set, that is $\tilde{E}(e) = \tilde{1}$ for all $e \in E$,

and $\tilde{C} = M_2(R(C)^*)$, be the C^*-algebra.

Define $\tilde{d}_{c^*}\colon \tilde{E} \times \tilde{E} \to \tilde{C}$ by

$\tilde{d}_{c^*}(F_{e_1}, F_{e_2}) = (Inf\{|F_{e_1}(a) - F_{e_2}(a)|/a \in C\}, 0)$,

then obviously $(\tilde{E}, \tilde{C}, \tilde{d}_{c^*})$ is a complete C^*-algebra valued fuzzy soft metric space.

We define $S\colon \tilde{E} \times \tilde{E} \to \tilde{E}$ by $S(F_{e_1}, G_{e_1})(a) = \frac{F_{e_1}^2 + G_{e_1}^2}{4}$,

$f\colon \tilde{E} \to \tilde{E}$ by $fF_{e_1} = \frac{F_{e_1}}{3}$ and $g\colon \tilde{E} \to \tilde{E}$ by $gF_{e_1} = \frac{F_{e_1}}{2}$ for all $a \in U$ and $F_{e_1}, G_{e_1} \in \tilde{E}$. Notice that, $fF_{e_1} = \frac{F_{e_1}}{3} = \{0.10, 0.06, 0.13, 0.03\}$ and $gF_{e_2} = \frac{F_{e_1}}{2} = \{0.25, 0.2, 0.3, 0.15\}$.

Thus,

$Inf\{|\mu^a_{fF_{e_1}}(s) - \mu^a_{gF_{e_2}}(s)|/s \in C\} = Inf\{0.15, 0.14, 0.17, 0.12\} = 0.12$

Hence $\tilde{d}_{c^*}(fF_{e_1}, gF_{e_2}) = \begin{bmatrix} 0.12 & 0 \\ 0 & 0.12 \end{bmatrix}$

also, $fG_{e_1} = \frac{G_{e_1}}{3} = \{0.13, 0.2, 0.1, 0.06\}$ and

$gG_{e_2} = \frac{G_{e_1}}{2} = \{0.4, 0.45, 0.25, 0.35\}$.

Thus,

$Inf\{|\mu^a_{fG_{e_1}}(s) - \mu^a_{gG_{e_2}}(s)|/s \in C\} = Inf\{0.27, 0.25, 0.15, 0.29\} = 0.15$

and $\tilde{d}_{c^*}(fG_{e_1}, gG_{e_2}) = \begin{bmatrix} 0.15 & 0 \\ 0 & 0.15 \end{bmatrix}$

Moreover, $S(F_{e_1}, G_{e_1})(a) = \frac{F_{e_1}^2 + G_{e_1}^2}{4} = \{0.062, 0.1, 0.062, 0.012\}$

and $S(F_{e_2}, G_{e_2})(a) = \frac{F_{e_2}^2 + G_{e_2}^2}{4} = \{0.222, 0.242, 0.152, 0.145\}$

Then

$$\begin{aligned} &\tilde{d}_{c^*}(S(F_{e_1}, G_{e_1}), S(F_{e_2}, G_{e_2})) \\ &= \begin{bmatrix} 0.09 & 0 \\ 0 & 0.09 \end{bmatrix} \\ &\preceq \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} 0.27 & 0 \\ 0 & 0.27 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \end{aligned}$$

$$\preceq \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \left(\begin{bmatrix} 0.12 & 0 \\ 0 & 0.12 \end{bmatrix} + \begin{bmatrix} 0.15 & 0 \\ 0 & 0.15 \end{bmatrix} \right) \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} \end{bmatrix}$$

$$\preceq \tilde{c}^* \left(\tilde{d}_{c^*}(fF_{e_1}, gF_{e_2}) + \tilde{d}_{c^*}(fG_{e_1}, gG_{e_2}) \right) \tilde{c}.$$

Here $\tilde{c} = \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} \end{bmatrix}$ with $||\tilde{c}|| = \frac{1}{\sqrt{3}} < \frac{1}{\sqrt{2}}$

Therefore, all the conditions of Theorem (3.1.4) satisfied.

Hence S, f and g have a unique coupled fixed point.

Theorem 3.1.7. Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space.

Suppose $S\colon \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f, g\colon \tilde{E} \to \tilde{E}$ be satisfying

(3.1.7.1) $S(\tilde{E} \times \tilde{E}) \subseteq g(\tilde{E})$ and $S(\tilde{E} \times \tilde{E}) \subseteq f(\tilde{E})$,

(3.1.7.2) $\{S, f\}$ and $\{S, g\}$ are ω-compatible pairs,

(3.1.7.3) one of $f(\tilde{E})$ or $g(\tilde{E})$ is complete C^*-algebra valued fuzzy soft metrics of $\tilde{E}$,

(3.1.7.4) $\tilde{d_{c^*}}\,(S(F_{e_1}, G_{e_1}), S(F_{e_2}, G_{e_2}))$

$$\preceq \tilde{a}\tilde{d_{c^*}}(S(F_{e_1}, G_{e_1}), fF_{e_1}) + \tilde{a}\tilde{d_{c^*}}(S(F_{e_2}, G_{e_2}), gF_{e_2})$$

for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$,

where $\tilde{a} \in \tilde{C}$ with $||\tilde{a}|| < \frac{1}{2}$. Then S, f and g have a unique common coupled fixed point in $\tilde{E} \times \tilde{E}$. Moreover, S, f and g have a unique common fixed point in $\tilde{E}$.

Proof: Similar to Theorem 3.1.4, construct four sequences $\{F_{e_{2n}}\}$, $\{G_{e_{2n}}\}$,

$\{I_{e_{2n}}\}$, $\{J_{e_{2n}}\}$ in $\tilde{E}$ such that

$$S(F_{e_{2n}}, G_{e_{2n}}) = fF_{e_{2n+1}} = I_{e_{2n}} \quad S(F_{e_{2n+1}}, G_{e_{2n+1}}) = gF_{e_{2n+2}} = I_{e_{2n+1}}$$
$$S(G_{e_{2n}}, F_{e_{2n}}) = fG_{e_{2n+1}} = J_{e_{2n}} \quad S(G_{e_{2n+1}}, F_{e_{2n+1}}) = gG_{e_{2n+2}} = J_{e_{2n+1}}$$
$$for \quad n = 0, 1, 2, \ldots$$

From (3.1.7.4), we have

$$\begin{aligned} \tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n+2}}) &= \tilde{d_{c^*}}\left(S(F_{e_{2n+1}}, G_{e_{2n+1}}), S(F_{e_{2n+2}}, G_{e_{2n+2}})\right) \\ &\preceq \tilde{a}\tilde{d_{c^*}}(S(F_{e_{2n+1}}, G_{e_{2n+1}}), fF_{e_{2n+1}}) + \tilde{a}\tilde{d_{c^*}}(S(F_{e_{2n+2}}, G_{e_{2n+2}}), gF_{e_{2n+2}}) \\ &\preceq \tilde{a}\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n}}) + \tilde{a}\tilde{d_{c^*}}(I_{e_{2n+2}}, I_{e_{2n+1}}). \end{aligned}$$

Therefore, $(\tilde{I}_{\tilde{C}} - \tilde{a})\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n+2}}) \preceq \tilde{a}\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n}})$

and similarly, $(\tilde{I}_{\tilde{C}} - \tilde{a})\tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n+2}}) \preceq \tilde{a}\tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n}})$.

Since $a \in \tilde{C}'_+$ with $||\tilde{a}|| < \frac{1}{2}$, we have $\tilde{I}_{\tilde{C}} - \tilde{a}$ is invertible and

$(\tilde{I}_{\tilde{C}} - \tilde{a})^{-1}\tilde{a} \in \tilde{C}'_+$.

Therefore,

$$\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n+2}}) \preceq (\tilde{I}_{\tilde{C}} - \tilde{a})^{-1}\tilde{a}\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n}})$$
$$\tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n+2}}) \preceq (\tilde{I}_{\tilde{C}} - \tilde{a})^{-1}\tilde{a}\tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n}}).$$

Then

$$||\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n+2}})|| \leq ||(\tilde{I}_{\tilde{C}} - \tilde{a})^{-1}\tilde{a}||||\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n}})||$$
$$||\tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n+2}})|| \leq ||(\tilde{I}_{\tilde{C}} - \tilde{a})^{-1}\tilde{a}||||\tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n}})||.$$

It follows from the fact

$$||(\tilde{I}_{\tilde{C}} - \tilde{a})^{-1}\tilde{a}|| \leq ||(\tilde{I}_{\tilde{C}} - \tilde{a})^{-1}||||\tilde{a}|| \leq \sum_{m=o}^{\infty} ||\tilde{a}||^m||\tilde{a}|| = \frac{||\tilde{a}||}{1-||\tilde{a}||} < 1$$

that is $\{I_{e_{2n}}\}$, $\{J_{e_{2n}}\}$ are Cauchy sequences in $\tilde{E}$ with respect to $\tilde{C}$. It follows that $\{I_{e_{2n+1}}\}$ and $\{J_{e_{2n+1}}\}$ are also Cauchy sequences in $\tilde{E}$ with respect to $\tilde{C}$. Thus $\{I_{e_n}\}$ and $\{J_{e_n}\}$ are Cauchy sequences in $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$.

Suppose $g(\tilde{E})$ is complete subspace of $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$.

Then the sequences $\{I_{e_n}\}$ and $\{J_{e_n}\}$ are converge to $I_{e'}, J_{e'}$ respectively in $g(\tilde{E})$. Thus there exist $F_{e'}, G_{e'}$ in $g(\tilde{E})$.

Such that

$$\lim_{n\to\infty} I_{e_n} = I_{e'} = gF_{e'} \text{ and } \lim_{n\to\infty} J_{e_n} = J_{e'} = gG_{e'}.$$

We now claim that $S(F_{e'}, G_{e'}) = I_{e'}$ and $S(G_{e'}, F_{e'}) = J_{e'}$.

From (3.1.7.4) and using the triangular inequality, we have

$$\begin{aligned}
\tilde{0}_{\tilde{C}} &\preceq \tilde{d_{c^*}}(I_{e'}, S(F_{e'}, G_{e'})) \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{d_{c^*}}(I_{e_{2n+1}}, S(F_{e'}, G_{e'})) \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{d_{c^*}}(S(F_{e_{2n+1}}, G_{e_{2n+1}}), S(F_{e'}, G_{e'})) \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{a}\tilde{d_{c^*}}(S(F_{e_{2n+1}}, G_{e_{2n+1}}), fF_{e_{2n+1}}) + \tilde{a}\tilde{d_{c^*}}(S(F_{e'}, G_{e'}), gF_{e'}) \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{a}\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n}}) + \tilde{a}\tilde{d_{c^*}}(S(F_{e'}, G_{e'}), I_{e'}) \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{a}\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n}}) + \tilde{a}\tilde{d_{c^*}}(I_{e'}, S(F_{e'}, G_{e'})).
\end{aligned}$$

Which implies that

$$\tilde{d_{c^*}}(S(F_{e'}, G_{e'}), I_{e'}) \preceq (\tilde{I}_{\tilde{C}} - \tilde{a})^{-1}\tilde{a}\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n}}) + (\tilde{I}_{\tilde{C}} - \tilde{a})^{-1}\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e'}).$$

Taking the limit as $n \to \infty$ in the above relation, we obtain

$\tilde{d_{c^*}}(S(F_{e'}, G_{e'}), I_{e'}) = \tilde{0}_{\tilde{C}}$ and hence $S(F_{e'}, G_{e'}) = I_{e'}$.

Similarly, we prove $S(G_{e'}, F_{e'}) = J_{e'}$.

Therefore, it follows $S(F_{e'}, G_{e'}) = I_{e'} = gI_{e'}$ and $S(G_{e'}, F_{e'}) = J_{e'} = gJ_{e'}$.

Since $\{S, g\}$ is ω-compatible pair, we have $S(I_{e'}, J_{e'}) = gI_{e'}$

and $S(J_{e'}, I_{e'}) = gJ_{e'}$.

Now to prove that $gI_{e'} = I_{e'}$ and $gJ_{e'} = J_{e'}$.

From (3.1.7.4) and using the triangular inequality, we have

$$\begin{aligned}
\tilde{0}_{\tilde{C}} &\preceq \tilde{d_{c^*}}(I_{e'}, gI_{e'}) \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{d_{c^*}}(I_{e_{2n+1}}, gI_{e'}) \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{d_{c^*}}(S(F_{e_{2n+1}}, G_{e_{2n+1}}), S(I_{e'}, J_{e'})) \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{a}\tilde{d_{c^*}}(S(F_{e_{2n+1}}, G_{e_{2n+1}}), fF_{e_{2n+1}}) + \tilde{a}\tilde{d_{c^*}}(S(I_{e'}, J_{e'}), gI_{e'}) \\
&\preceq \tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + \tilde{a}\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n}}) + \tilde{a}\tilde{d_{c^*}}(I_{e'}, gI_{e'}).
\end{aligned}$$

Which implies

$$\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(I_{e'}, gI_{e'}) \preceq (\tilde{I}_{\tilde{C}} - \tilde{a})^{-1}\tilde{d_{c^*}}(I_{e'}, I_{e_{2n+1}}) + (\tilde{I}_{\tilde{C}} - \tilde{a})^{-1}\tilde{a}\tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n}}).$$

Taking the limit as $n \to \infty$ in the above relation, we obtain

$\tilde{d_{c^*}}(I_{e'}, gI_{e'}) = 0_{\tilde{C}}$ which implies $gI_{e'} = I_{e'}$. Similarly we can prove $gJ_{e'} = J_{e'}$.

Therefore, $S(I_{e'}, J_{e'}) = gI_{e'} = I_{e'}$ and $S(J_{e'}, I_{e'}) = gJ_{e'} = J_{e'}$.

Thus $(I_{e'}, J_{e'})$ is common coupled fixed point of S and g.

Since $S(\tilde{E} \times \tilde{E}) \subseteq f(\tilde{E})$. So there exist $K_{e'}$, $L_{e'} \in \tilde{E}$ such that

$S(I_{e'}, J_{e'}) = I_{e'} = fK_{e'}$ and $S(J_{e'}, I_{e'}) = J_{e'} = fL_{e'}$.

Now from (3.1.7.4) and using the triangular inequality, we have

$$\begin{aligned}
\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(S(K_{e'}, L_{e'}), I_{e'}) &\preceq \tilde{d_{c^*}}(S(K_{e'}, L_{e'}), S(I_{e'}, J_{e'})) \\
&\preceq \tilde{a}\tilde{d_{c^*}}(S(K_{e'}, L_{e'}), fK_{e'}) + \tilde{a}\tilde{d_{c^*}}(S(I_{e'}, J_{e'}), gI_{e'}) \\
&\preceq \tilde{a}\tilde{d_{c^*}}(S(K_{e'}, L_{e'}), I_{e'}) + \tilde{a}\tilde{d_{c^*}}((I_{e'}, I_{e'}).
\end{aligned}$$

We have $\tilde{d_{c^*}}(S(K_{e'}, L_{e'}), I_{e'}) = 0$, which means that $I_{e'} = S(K_{e'}, L_{e'})$.

Similarly, we have $S(L_{e'}, K_{e'}) = J_{e'}$.

Since $\{S, f\}$ is ω-compatible pair, we have $S(I_{e'}, J_{e'}) = fI_{e'}$

and $S(J_{e'}, I_{e'}) = fJ_{e'}$.

Now we prove that $fI_{e'} = I_{e'}$ and $fJ_{e'} = J_{e'}$.

From (3.1.7.4) and using the triangular inequality, we have

$$\begin{aligned}\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(I_{e'}, fI_{e'}) &\preceq \tilde{d_{c^*}}(S(I_{e'}, J_{e'}), S(I_{e'}, J_{e'}))\\ &\preceq \tilde{a}\tilde{d_{c^*}}(S(I_{e'}, J_{e'}), fI_{e'}) + \tilde{a}\tilde{d_{c^*}}(S(I_{e'}, J_{e'}), gI_{e'})\\ &\preceq \tilde{a}\tilde{d_{c^*}}(I_{e'}, fI_{e'}) + \tilde{a}\tilde{d_{c^*}}(gI_{e'}, gI_{e'})\\ &\preceq \tilde{a}\tilde{d_{c^*}}(I_{e'}, fI_{e'})\end{aligned}$$

which means that $fI_{e'} = I_{e'}$ and

$$\begin{aligned}\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(J_{e'}, fJ_{e'}) &\preceq \tilde{d_{c^*}}(S(J_{e'}, I_{e'}), S(J_{e'}, I_{e'}))\\ &\preceq \tilde{a}\tilde{d_{c^*}}(S(J_{e'}, I_{e'}), fJ_{e'}) + \tilde{a}\tilde{d_{c^*}}(S(J_{e'}, I_{e'}), gJ_{e'})\\ &\preceq \tilde{a}\tilde{d_{c^*}}(J_{e'}, fJ_{e'}) + \tilde{a}\tilde{d_{c^*}}(gJ_{e'}, gJ_{e'})\\ &\preceq \tilde{a}\tilde{d_{c^*}}(J_{e'}, fJ_{e'})\end{aligned}$$

which means that $fJ_{e'} = J_{e'}$. Therefore, we have $S(I_{e'}, J_{e'}) = fI_{e'} = I_{e'}$ and $S(J_{e'}, I_{e'}) = fJ_{e'} = J_{e'}$.

Thus $(I_{e'}, J_{e'})$ is common coupled fixed point of S, f and g.

The same reasoning that in Theorem 3.1.4 tells us that $I_{e'} = J_{e'}$, which means that S, f and g have a unique fixed point in $\tilde{E}$.

Application to the existence of solutions of integral equations

Theorem 3.1.8. Let us Consider the integral equation

$$F_{e_1}(t) = \int_C (K_1(t, s, F_{e_1}(s)) + K_2(t, s, F_{e_1}(s)))\, ds, t \in C$$

Where C is a Lebesgue measurable set. Suppose that

(i) $K_1, K_2 : C \times C \times R(C)^* \to R(C)^*$,

(ii) there exist two continuous function $\tau : C \times C \to R(C)^*$ and $r \in (0, 1)$ such that for $u, v \in C$ *and* $F_{e_1}(v), F_{e_2}(v) \in R(C)^*$,

$$\begin{aligned}&\inf\{|K_1(u, v, F_{e_1}(v)) - K_1(u, v, F_{e_2}(v))|\}\\ &\qquad \le r \inf\{|\tau(u, v)|\}. \inf\{|(F_{e_1}(v) - F_{e_2}(v))|\},\end{aligned}$$

(iii) $\sup\limits_{t\in C}\int\limits_{C}\inf\{|\tau(u,v)|\}dv \leq 1.$

then the integral equation has a unique solutions in $L^{\infty}(C)$.

Proof: Let $E = C = [0,1]$ and $\tilde{E} = L^{\infty}(C)$ be the set of essential bounded measuarble function on C and $H = L^{2}(C)$, where the parameter set C is a lebesgue measureable set. By $L(H)$ we denote the set of bounded linear operators on hilbert space H.

Consider $\tilde{d_{c^*}} : \tilde{E}\times\tilde{E} \to L(H)$ by $\tilde{d_{c^*}}(F_{e_1},F_{e_2}) = \mathrm{M}_{\inf\{|\mu^a_{F_{e_1}}(s)-\mu^a_{F_{e_2}}(s)|/s\in C\}}$ for all $F_{e_1},F_{e_2}\in\tilde{E}$, where $\mathrm{M}_h : H \to H$ is the multiplication operator defined by $\mathrm{M}_h(\tau) = h.\ \tau$ for $\tau \in H$. Then $\tilde{d_{c^*}}$ is a C^* - algebra valued fuzzy soft metric and $(\tilde{E}, L(H), \tilde{d_{c^*}})$ is a complete C^* - algebra valued fuzzy soft metric space.

Define two self mappings $S : \tilde{E}\times\tilde{E} \to \tilde{E}$ by

$$S(F_{e_1},G_{e_1})(t) = \int\limits_{C}\left(K_1(t,s,F_{e_1}(s)) + K_2(t,s,G_{e_1}(s))\right)ds, t\in C$$

Notice that

$$\tilde{d}_{c^*}(S(F_{e_1},G_{e_1}),S(F_{e_2},G_{e_2})) = M_{\inf\left\{\left|\mu^a_{S(F_{e_1},G_{e_1})}(s)-\mu^a_{S(F_{e_2},G_{e_2})}(s)\right|/s\in C\right\}}.$$

Now consider norm

$$\begin{aligned}
&\left\|\tilde{d}_{c^*}(S(F_{e_1},G_{e_1}),S(F_{e_2},G_{e_2}))\right\| \\
&= \underset{h=1}{Sup}\left(M_{\inf\left\{\left|\mu^a_{S(F_{e_1},G_{e_1})}(s)-\mu^a_{S(F_{e_2},G_{e_2})}(s)\right|/s\in C\right\}}h,h\right) \\
&= \underset{\|h\|=1}{Sup}\int\limits_{C}\left[\inf\left\{\left|\mu^a_{S(F_{e_1},G_{e_1})}(s)-\mu^a_{S(F_{e_2},G_{e_2})}(s)\right|/s\in C\right\}\right]h(t)\overline{h(t)}dt \\
&\leq \underset{\|h\|=1}{Sup}\int\limits_{C}\left[\int\limits_{C}\inf\{|K_1(t,s,F_{e_1}(s)) - K_1(t,s,F_{e_2}(s))|\}ds\right]|h(t)|^2\,dt \\
&\quad + \underset{\|h\|=1}{Sup}\int\limits_{C}\left[\int\limits_{C}\inf\{|K_1(t,s,G_{e_1}(s)) - K_1(t,s,G_{e_2}(s))|\}ds\right]|h(t)|^2\,dt \\
&\leq \underset{\|h\|=1}{Sup}\int\limits_{C}\left[\int\limits_{C}r\inf\{|\tau(t,s)(F_{e_1}(s)) - \tau(t,s)(F_{e_2}(s))|\}ds\right]|h(t)|^2\,dt \\
&\quad + \underset{\|h\|=1}{Sup}\int\limits_{C}\left[\int\limits_{C}r\inf\{|\tau(t,s)(G_{e_1}(s)) - \tau(t,s)(G_{e_2}(s))|\}ds\right]|h(t)|^2\,dt
\end{aligned}$$

$$
\begin{aligned}
&\leq r \underset{\|h\|=1}{Sup} \int_C \left[\int_C \inf\{|\tau(t,s)|\} \inf\{|F_{e_1}(s) - F_{e_2}(s)|\} ds \right] |h(t)|^2 \, dt \\
&\quad + r \underset{\|h\|=1}{Sup} \int_C \left[\int_C \inf\{|\tau(t,s)|\} \inf\{|G_{e_1}(s) - G_{e_2}(s)|\} ds \right] |h(t)|^2 \, dt \\
&\leq r \underset{\|h\|=1}{Sup} \int_C \left[\int_C \inf\{|\tau(t,s)|\} ds \right] |h(t)|^2 \, dt. \|\inf\{|F_{e_1}(s) - F_{e_2}(s)|\}\|_\infty \\
&\quad + r \underset{\|h\|=1}{Sup} \int_C \left[\int_C \inf\{|\tau(t,s)|\} ds \right] |h(t)|^2 \, dt. \|\inf\{|G_{e_1}(s) - G_{e_2}(s)|\}\|_\infty \\
&\leq r \underset{\|h\|=1}{Sup} \int_C \inf\{|\tau(t,s)|\} ds. \underset{\|h\|=1}{Sup} \int_C |h(t)|^2 \, dt. \|\inf\{|F_{e_1}(s) - F_{e_2}(s)|\}\|_\infty \\
&\quad + r \underset{\|h\|=1}{Sup} \int_C \inf\{|\tau(t,s)|\} ds. \underset{\|h\|=1}{Sup} \int_C |h(t)|^2 \, dt. \|\inf\{|G_{e_1}(s) - G_{e_2}(s)|\}\|_\infty \\
&\leq r \|\inf\{|F_{e_1}(s) - F_{e_2}(s)|\}\|_\infty + r \|\inf\{|G_{e_1}(s) - G_{e_2}(s)|\}\|_\infty .
\end{aligned}
$$

Set $\tilde{a} = \sqrt{r} 1_{L(H)}$, then $\tilde{a} \in L(H)$ and $||\tilde{a}|| = \sqrt{r} < \frac{1}{\sqrt{2}}$. Hence, applying our corollary(3.1.5) , we get the desired result.

SECTION 3.2 : COINCIDENCE POINT THEOREM BY USING HYBRID PAIROF MAPPINGS IN C^*-ALGEBRA VALUED FUZZY SOFT METRIC SPACES

In this section, we establish a coincidence point theorem for a hybrid pair of single valued and multivalued mappings in complete C^*-algebra valued fuzzy soft metric spaces. An example is also given to validate our result.

We need the following definitions and results in the sequel.

Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space. We denote by $CB(\tilde{E})$ be a class of all non-empty closed and bounded subsets of $\tilde{E}$. For the points $F_{e_1}, F_{e_2} \in \tilde{E}$ and $\tilde{X}, \tilde{Y} \in CB(\tilde{E})$,

define $\tilde{D_{c^*}}(F_{e_1}, \tilde{Y}) = \inf_{G_{e_1} \in \tilde{Y}} \tilde{d_{c^*}}(F_{e_1}, G_{e_1})$.

Let $\tilde{H_{c^*}}$ be the Hausdorff C^*-algebra valued fuzzy soft metric induced by the C^*-algebra valued fuzzy soft metric $\tilde{d_{c^*}}$ on $\tilde{E}$, that is

$$\tilde{H_{c^*}}(\tilde{X}, \tilde{Y}) = \max\left\{ \sup_{F_{e_1} \in \tilde{X}} \tilde{D_{c^*}}(F_{e_1}, \tilde{Y}), \sup_{G_{e_1} \in \tilde{Y}} \tilde{D_{c^*}}(\tilde{X}, G_{e_1}) \right\}$$

for every $\tilde{X}, \tilde{Y} \in CB(\tilde{E})$. It is well known that $\left(CB(\tilde{E}), \tilde{C}, \tilde{H_{c^*}}\right)$ is a complete C^*-algebra valued fuzzy soft metric space, whenever $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ is a complete C^*-algebra valued fuzzy soft metric space.

Definition 3.2.1.(R.P.Agarwal et al.[79]): Let $T : \tilde{E} \to CB(\tilde{E})$ be a multi-valued map. An element $F_{e_1} \in \tilde{E}$ is fixed point of T if $F_{e_1} \in TF_{e_1}$

Definition 3.2.2.(R.P.Agarwal et al.[79]): Let $T : \tilde{E} \to CB(\tilde{E})$ and $f : \tilde{E} \to \tilde{E}$ be a multi-valued map and single valued maps. An element $F_{e_1} \in \tilde{E}$ is coincidence point of T and f if $fF_{e_1} \in TF_{e_1}$. We denote

$$C\{f, T\} = \left\{ F_{e_1} \in \tilde{E} / fF_{e_1} \in TF_{e_1} \right\}$$

Definition 3.2.3.(R.P.Agarwal et al.[79]): An element $F_{e_1} \in \tilde{E}$ is a common fixed point of $T : \tilde{E} \to CB(\tilde{E})$and $f : \tilde{E} \to \tilde{E}$ if $F_{e_1} = fF_{e_1} \in TF_{e_1}$.

Theorem 3.2.4. Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space, and $T \colon \tilde{E} \to CB(\tilde{E})$ be a multi-valued map satisfying

$$\tilde{H_{c^*}}\,(TF_{e_1}, TF_{e_2}) \preceq \tilde{a}^{\star}\tilde{d_{c^*}}(F_{e_1}, F_{e_2})\tilde{a} \tag{1}$$

for all $F_{e_1}, F_{e_2} \in \tilde{E}$, where $\tilde{a} \in \tilde{C}$ with $||\tilde{a}|| < 1$. Then T has a unique fixed point in $\tilde{E}$.

Lemma 3.2.5. If $\tilde{X}, \tilde{Y} \in CB(\tilde{E})$ and $F_{e_1} \in \tilde{X}$, then for any fixed $\tilde{b} \in \tilde{C_+}'$ with $||\tilde{b}|| < 1$, there exist $F_{e_2} = F_{e_2}(F_{e_1}) \in \tilde{Y}$ such that

$$\tilde{d_{c^*}}\,(F_{e_1}, F_{e_2}) \preceq \tilde{b}\tilde{H_{c^*}}(\tilde{X}, \tilde{Y}) \tag{2}$$

Now we give our main result.

Theorem 3.2.6. Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space. Let $S \colon \tilde{E} \to CB(\tilde{E})$ be a multi-valued map and $f : \tilde{E} \to \tilde{E}$ be a single-valued map. Suppose that

$$\begin{aligned} \tilde{H_{c^*}}\,(SF_{e_1}, SF_{e_2}) \;\preceq\; & \tilde{a}\tilde{d_{c^*}}(fF_{e_1}, fF_{e_2}) \\ & +\tilde{a}\left(\tilde{D_{c^*}}(fF_{e_1}, SF_{e_1}) + \tilde{D_{c^*}}(fF_{e_2}, SF_{e_2})\right) \\ & +\tilde{a}\left(\tilde{D_{c^*}}(fF_{e_1}, SF_{e_2}) + \tilde{D_{c^*}}(fF_{e_2}, SF_{e_1})\right) \end{aligned} \tag{3}$$

for all $F_{e_1}, F_{e_2} \in \tilde{E}$, where $\tilde{a} \in \tilde{C_+}'$ with $||\tilde{a}|| < 1$. Suppose that

(A_1) $S(\tilde{E}) \subseteq f(\tilde{E})$;

(A_2) $f(\tilde{E})$ is closed.

Then, there exists point $F'_e \in \tilde{E}$, such that $fF'_e \in SF'_e$.

Proof: Let $F_{e_0} \in \tilde{E}$ be arbitrary. Then, fF_{e_0} and SF_{e_0} are well defined. From (A_1), there exists $F_{e_1} \in \tilde{E}$, such that $fF_{e_1} \in SF_{e_0}$.
Again from (A_1) and Lemma 3.2.5 with $||\tilde{b}|| < 1$, as $fF_{e_1} \in SF_{e_0}$, there exists $F_{e_2} \in \tilde{E}$ such that $fF_{e_2} \in SF_{e_1}$ and

$$\tilde{d_{c^*}}(fF_{e_1}, fF_{e_2}) \preceq \tilde{b}\tilde{H_{c^*}}(SF_{e_0}, SF_{e_1}) \tag{4}$$

from (3) and (4), we get

$$\begin{aligned}
\tilde{d_{c^*}}(fF_{e_1}, fF_{e_2}) &\preceq \tilde{b}\tilde{H_{c^*}}(SF_{e_0}, SF_{e_1}) \\
&\preceq \tilde{b}\tilde{a}\tilde{d_{c^*}}(fF_{e_0}, fF_{e_1}) \\
&\quad + \tilde{b}\tilde{a}\left(\tilde{D_{c^*}}(fF_{e_0}, SF_{e_0}) + \tilde{D_{c^*}}(fF_{e_1}, SF_{e_1})\right) \\
&\quad + \tilde{b}\tilde{a}\left(\tilde{D_{c^*}}(fF_{e_0}, SF_{e_1}) + \tilde{D_{c^*}}(fF_{e_1}, SF_{e_0})\right).
\end{aligned} \tag{5}$$

In contrast, we have

$$\begin{aligned}
\tilde{D_{c^*}}(fF_{e_0}, SF_{e_0}) &\preceq \tilde{d_{c^*}}(fF_{e_0}, fF_{e_1}) \\
\tilde{D_{c^*}}(fF_{e_1}, SF_{e_1}) &\preceq \tilde{d_{c^*}}(fF_{e_1}, fF_{e_2}) \\
\tilde{D_{c^*}}(fF_{e_1}, SF_{e_0}) &\preceq \tilde{d_{c^*}}(fF_{e_1}, fF_{e_1}) = 0 \\
\tilde{D_{c^*}}(fF_{e_0}, SF_{e_1}) &\preceq \tilde{d_{c^*}}(fF_{e_0}, fF_{e_2}) \\
&\preceq \tilde{d_{c^*}}(fF_{e_0}, fF_{e_1}) + \tilde{d_{c^*}}(fF_{e_1}, fF_{e_2})
\end{aligned} \tag{6}$$

from (5) and (6), we get

$$\begin{aligned}
\tilde{d_{c^*}}(fF_{e_1}, fF_{e_2}) &\preceq \tilde{b}\tilde{a}\tilde{d_{c^*}}(fF_{e_0}, fF_{e_1}) \\
&\quad + \tilde{b}\tilde{a}\left(\tilde{d_{c^*}}(fF_{e_0}, fF_{e_1}) + \tilde{d_{c^*}}(fF_{e_1}, fF_{e_2})\right) \\
&\quad + \tilde{b}\tilde{a}\left(\tilde{d_{c^*}}(fF_{e_0}, fF_{e_1}) + \tilde{d_{c^*}}(fF_{e_1}, fF_{e_2})\right) \\
&= 3\ \tilde{b}\tilde{a}\tilde{d_{c^*}}(fF_{e_0}, fF_{e_1}) + 2\tilde{b}\tilde{a}\tilde{d_{c^*}}(fF_{e_1}, fF_{e_2})
\end{aligned} \tag{7}$$

Therefore,

$$(1-2\tilde{b}\tilde{a})\tilde{d_{c^*}}(fF_{e_1}, fF_{e_2}) \preceq 3\tilde{b}\tilde{a}\tilde{d_{c^*}}(fF_{e_0}, fF_{e_1})$$

Since $||\tilde{b}||||\tilde{a}|| < \frac{1}{2}$, we have $1-2\tilde{b}\tilde{a}$ is invertible, and can expressed as $(1-2\tilde{b}\tilde{a})^{-1} = \sum_{m=0}^{\infty}(2\tilde{b}\tilde{a})^m$, which together with $2\tilde{b}\tilde{a} \in \tilde{C_+}'$
can yields $(1-2\tilde{b}\tilde{a})^{-1} \in \tilde{C_+}'$. By Lemma 1.6.7(iii)(Ch-1), we know

$$\tilde{d_{c^*}}(fF_{e_1}, fF_{e_2}) \preceq \tilde{\kappa}\tilde{d_{c^*}}(fF_{e_0}, fF_{e_1})$$

where $\tilde{\kappa} = 3\tilde{b}\tilde{a}(1-2\tilde{b}\tilde{a})^{-1} \in \tilde{C_+}'$ with $||3\tilde{b}\tilde{a}(1-2\tilde{b}\tilde{a})^{-1}|| < 1$.

Again from (A_1) and Lemma 3.2.5 with $||\tilde{b}|| < 1$, as $fF_{e_2} \in SF_{e_1}$, there exists $F_{e_3} \in \tilde{E}$ such that $fF_{e_3} \in SF_{e_2}$ and

$$\tilde{d_{c^*}}(fF_{e_2}, fF_{e_3}) \preceq \tilde{b}\tilde{H_{c^*}}(SF_{e_2}, SF_{e_1}) \tag{8}$$

from (3) and (8), we get

$$\begin{aligned}
\tilde{d_{c^*}}(fF_{e_2}, fF_{e_3}) &\preceq \tilde{b}\tilde{H_{c^*}}(SF_{e_2}, SF_{e_1}) \\
&\preceq \tilde{b}\tilde{a}\tilde{d_{c^*}}(fF_{e_2}, fF_{e_1}) \\
&\quad +\tilde{b}\tilde{a}\left(\tilde{D_{c^*}}(fF_{e_2}, SF_{e_2}) + \tilde{D_{c^*}}(fF_{e_1}, SF_{e_1})\right) \\
&\quad +\tilde{b}\tilde{a}\left(\tilde{D_{c^*}}(fF_{e_2}, SF_{e_1}) + \tilde{D_{c^*}}(fF_{e_1}, SF_{e_2})\right).
\end{aligned} \tag{9}$$

In contrast, we have

$$\begin{aligned}
&\tilde{D_{c^*}}(fF_{e_2}, SF_{e_2}) \preceq \tilde{d_{c^*}}(fF_{e_2}, fF_{e_3}) \\
&\tilde{D_{c^*}}(fF_{e_1}, SF_{e_1}) \preceq \tilde{d_{c^*}}(fF_{e_1}, fF_{e_2}) \\
&\tilde{D_{c^*}}(fF_{e_2}, SF_{e_1}) \preceq \tilde{d_{c^*}}(fF_{e_2}, fF_{e_2}) = 0 \\
&\tilde{D_{c^*}}(fF_{e_1}, SF_{e_2}) \preceq \tilde{d_{c^*}}(fF_{e_1}, fF_{e_3}) \\
&\qquad \preceq \tilde{d_{c^*}}(fF_{e_1}, fF_{e_2}) + \tilde{d_{c^*}}(fF_{e_2}, fF_{e_3}).
\end{aligned} \tag{10}$$

Similarly as above, from (9) and (10), we get

$$\tilde{d_{c^*}}\left(fF_{e_2}, fF_{e_3}\right) \preceq \tilde{\kappa}\tilde{d_{c^*}}(fF_{e_1}, fF_{e_2})$$

Continuing this process, we can construct a sequence $\{G_{e_n}\}$ in $\tilde{E}$, such that $G_{e_0} = fF_{e_1}$ and, for each $n \in N$,

$$G_{e_{2n}} = fF_{e_{2n+1}} \in SF_{e_{2n}} \qquad G_{e_{2n+1}} = fF_{e_{2n+2}} \in SF_{e_{2n+1}} \tag{11}$$

and

$$\tilde{d_{c^*}}\left(G_{e_{2n}}, G_{e_{2n+1}}\right) = \tilde{d_{c^*}}(fF_{e_{2n+1}}, fF_{e_{2n+2}}) \preceq \tilde{\kappa}\tilde{d_{c^*}}(fF_{e_{2n+1}}, fF_{e_{2n}})$$
$$\tilde{d_{c^*}}\left(G_{e_{2n-1}}, G_{e_{2n}}\right) = \tilde{d_{c^*}}(fF_{e_{2n}}, fF_{e_{2n+1}}) \preceq \tilde{\kappa}\tilde{d_{c^*}}(fF_{e_{2n-1}}, fF_{e_{2n}}).$$

Therefore, we have

$$\tilde{d_{c^*}}\left(G_{e_n}, G_{e_{n+1}}\right) \preceq \tilde{\kappa}\tilde{d_{c^*}}\left(G_{e_{n-1}}, G_{e_n}\right) \text{ for all} n \geq 1 \tag{12}$$

From (12), by induction and Lemma (1.6.7) (iii), we get

$$\tilde{d_{c^*}}\left(G_{e_n}, G_{e_{n+1}}\right) \preceq \tilde{\kappa}^n\tilde{d_{c^*}}\left(G_{e_0}, G_{e_1}\right) \text{ for all} n \in N \tag{13}$$

Now, we shall show that $\{G_{e_n}\}$ is a Cauchy sequence in $\tilde{E}$.

For $m > n$, by using triangle inequality and (13) we have

$$\begin{aligned}
\tilde{d_{c^*}}\left(G_{e_n}, G_{e_m}\right) &\preceq \tilde{d_{c^*}}\left(G_{e_n}, G_{e_{n+1}}\right) + \tilde{d_{c^*}}\left(G_{e_{n+1}}, G_{e_{n+2}}\right) + \cdots + \tilde{d_{c^*}}\left(G_{e_{m-1}}, G_{e_m}\right) \\
&\preceq \left(\tilde{\kappa}^n + \tilde{\kappa}^{n+1} + \tilde{\kappa}^{n+2} + \cdots + \tilde{\kappa}^{m-1}\right)\tilde{d_{c^*}}\left(G_{e_0}, G_{e_1}\right) \\
&\leq ||\tilde{\kappa}^n + \tilde{\kappa}^{n+1} + \tilde{\kappa}^{n+2} + \cdots + \tilde{\kappa}^{m-1}||||\tilde{d_{c^*}}\left(G_{e_0}, G_{e_1}\right)||I\tilde{C} \\
&\leq ||\tilde{\kappa}^n|| + ||\tilde{\kappa}^{n+1}|| + \cdots + ||\tilde{\kappa}^{m-1}||||\tilde{d_{c^*}}\left(G_{e_0}, G_{e_1}\right)||I\tilde{C} \\
&= \tfrac{||\tilde{\kappa}||^n}{1-||\tilde{\kappa}||}||\tilde{d_{c^*}}\left(G_{e_0}, G_{e_1}\right)||I\tilde{C} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence $\{G_{e_n}\}$ is a Cauchy sequence. Now as,$(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space, $\{G_{e_n}\}$ converges to some $G_{e'} \in \tilde{E}$.

Therefore,

$$\lim_{n\to\infty} G_{e_n} = \lim_{n\to\infty} fF_{e_{2n+1}} = G_{e'}. \tag{14}$$

As $G_{e_{2n}} = fF_{e2n+1}$ and $f(\tilde{E})$ is closed, we have $G_{e'} \in f(\tilde{E})$. Hence there exist $F_{e'} \in \tilde{E}$, such that $fF_{e'} = G_{e'}$. From the contraction type condition (3) and (11), we obtain

$$\begin{aligned}
\tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right) &\preceq \tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + \tilde{D_{c^*}}\left(fF_{e_{2n+2}}, SF_{e'}\right) \\
&\preceq \tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + \tilde{H_{c^*}}\left(SF_{e'}, SF_{e_{2n+1}},\right) \\
&\preceq \tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + \tilde{a}\tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+1}},\right) \\
&\quad +\tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right) + \tilde{D_{c^*}}\left(fF_{e_{2n+1}}, SF_{e_{2n+1}}\right)\right) \\
&\quad +\tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e'}, SF_{e_{2n+1}}\right) + \tilde{D_{c^*}}\left(fF_{e_{2n+1}}, SF_{e'}\right)\right) \\
&\preceq \tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + \tilde{a}\tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+1}},\right) \\
&\quad +\tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right) + \tilde{D_{c^*}}\left(fF_{e_{2n+1}}, fF_{e_{2n+2}}\right)\right) \\
&\quad +\tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + \tilde{D_{c^*}}\left(fF_{e_{2n+1}}, SF_{e'}\right)\right)
\end{aligned}$$

which implies

$$\begin{aligned}
\tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right) &\preceq (1-\tilde{a})^{-1}\tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + (1-\tilde{a})^{-1}\tilde{a}\tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+1}},\right) \\
&\quad +(1-\tilde{a})^{-1}\tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e_{2n+1}}, fF_{e_{2n+2}}\right)\right) \\
&\quad +(1-\tilde{a})^{-1}\tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + \tilde{D_{c^*}}\left(fF_{e_{2n+1}}, SF_{e'}\right)\right)
\end{aligned}$$

Letting $n \to \infty$ in the above inequality and $||(1-\tilde{a})^{-1}\tilde{a}|| < 1$, using (14), we get $\tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right) = 0$. Hence, as $SF_{e'}$ is closed, $fF_{e'} \in SF_{e'}$

Now we give an example to illustrate our Theorem 3.2.6.

Example 3.2.7. Let $E = \{e_1, e_2, e_3, e_4\}, U = \{a, b, c, d\}$

and $C = \{e_1, e_2, e_3\}$ be a subset of E. Define fuzzy soft set as,

$$(F_E, C) = \left\{ \begin{array}{c} e_1 = \{a_{0.3}, b_{0.4}, c_{0.1}, d_{0.2}\}, e_2 = \{a_{0.6}, b_{0.7}, c_{0.5}, d_{0.4}\}, \\ e_3 = \{a_{0.8}, b_{0.9}, c_{0.6}, d_{0.7}\} \end{array} \right\}$$

$$F_{e_1} = \mu_{F_{e_1}} = \{a_{0.3}, b_{0.4}, c_{0.1}, d_{0.2}\}, F_{e_2} = \mu_{F_{e_2}} = \{a_{0.6}, b_{0.7}, c_{0.5}, d_{0.4}\}$$

$$F_{e_3} = \mu_{F_{e_3}} = \{a_{0.8}, b_{0.9}, c_{0.6}, d_{0.7}\}$$

and $FSC(F_E) = \{F_{e_1}, F_{e_2}, F_{e_3}\}$, let $\tilde{E}$ be absolute fuzzy soft set, that is $\tilde{E}(e) = \tilde{1}$ for all $e \in E$, and $\tilde{C} = M_2(R(C)^*)$, be the C^*-algebra. Define $\tilde{d}_{c^*} : \tilde{E} \times \tilde{E} \to \tilde{C}$ by $\tilde{d}_{c^*}(F_{e_1}, F_{e_2}) = (Inf\{|F_{e_1}(a) - F_{e_2}(a)|/a \in C\}, 0)$, then obviously $(\tilde{E}, \tilde{C}, \tilde{d}_{c^*})$ is a complete C^*-algebra valued fuzzy soft metric space. We define $S\colon \tilde{E} \to CB(\tilde{E})$ by $SF_{e_1}(a) = \frac{F_{e_1}^2}{2} + \frac{9}{50}$, $f\colon \tilde{E} \to \tilde{E}$ by $fF_{e_1} = F_{e_1}$ for all $a \in U$ and $F_{e_1} \in \tilde{E}$. Notice that,

$fF_{e_1} = F_{e_1} = \{0.3, 0.4, 0.1, 0.2\}$ and $fF_{e_2} = F_{e_2} = \{0.6, 0.7, 0.5, 0.4\}$. Thus,

$\inf\{|\mu^a_{fF_{e_1}}(s) - \mu^a_{fF_{e_2}}(s)|/s \in C\} = \inf\{0.3, 0.3, 0.2, 0.4\} = 0.2$.

Hence $\tilde{d}_{c^*}(fF_{e_1}, fF_{e_2}) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$.

Also, we have

$$\begin{aligned} \tilde{d}_{c^*}(SF_{e_1}, SF_{e_2})(a) &= (\inf\{|SF_{e_1}(a) - SF_{e_2}(a)|/a \in C\}, 0) \\ &= (\inf\{0.135, 0.165, 0.12, 0.06\}, 0) = \begin{bmatrix} 0.06 & 0 \\ 0 & 0.06 \end{bmatrix} \\ &\preceq \begin{bmatrix} 0.16 & 0 \\ 0 & 0.16 \end{bmatrix} \end{aligned}$$

$$\preceq \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$\preceq \tilde{c}\tilde{d_{c^*}}(fF_{e_1}, fF_{e_2})$$

Here $\tilde{c} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}$ with $||\tilde{c}|| = 0.8 < 1$. Therefore, (3) holds for all $F_{e_1}, F_{e_2} \in \tilde{E}$. Also, the other hypotheses (A_1) and (A_2) are satisfied. It is seen that $S(0.2) = f(0.2) = 0.2$. Therefore, S and f have the coincidence at the point $F_{e'} = 0.2$.

Corollary 3.2.8. Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space. Let $S\colon \tilde{E} \to CB(\tilde{E})$ be a pair of multivalued map. Suppose that

$$\begin{aligned} \tilde{H_{c^*}}(SF_{e_1}, SF_{e_2}) \preceq\ & \tilde{a}\tilde{d_{c^*}}(F_{e_1}, F_{e_2}) + \tilde{a}\left(\tilde{D_{c^*}}(F_{e_1}, SF_{e_1}) + \tilde{D_{c^*}}(F_{e_2}, SF_{e_2})\right) \\ & + \tilde{a}\left(\tilde{D_{c^*}}(F_{e_1}, SF_{e_2}) + \tilde{D_{c^*}}(F_{e_2}, SF_{e_1})\right) \qquad (15) \end{aligned}$$

for all $F_{e_1}, F_{e_2} \in \tilde{E}$, where $\tilde{a} \in \tilde{C_+}'$ with $||\tilde{a}|| < 1$. Then there exist a point $F_{e'} \in \tilde{E}$ such that $F_{e'} \in SF_{e'}$.

CHAPTER 4

UNIQUE COMMON FIXED POINT THEOREM FOR FOUR MAPS IN COMPLEX VALUED $S-$METRIC SPACES

In this chapter we obtain a common fixed point theorem for the two weakly compatible pairs of mappings satisfying a contractive condition in complex valued S-metric spaces. Also we give an example to illustrate our main theorem.

In 2016 Naval Singh et al.[68] proved the following theorem in complex valued metric spaces as follows.

Theorem 4.1.(Navalsingh et al.[68]): Let (X, d) be a complete complex valued metric space and $S, T : X \to X$. If $\exists$ mappings $\lambda, \mu, \gamma, \delta : X \times X \times X \to \mathbb{R}^+$ such that for all $x, y \in X$,

(a)$\lambda(TSx, y, a) \leq \lambda(x, y, a)$ and $\lambda(x, STy, a) \leq \lambda(x, y, a)$,

$\mu(TSx, y, a) \leq \mu(x, y, a)$ and $\mu(x, STy, a) \leq \mu(x, y, a)$,

$\gamma(TSx, y, a) \leq \gamma(x, y, a)$ and $\gamma(x, STy, a) \leq \gamma(x, y, a)$,

$\delta(TSx, y, a) \leq \delta(x, y, a)$ and $\delta(x, STy, a) \leq \delta(x, y, a)$,

(b)

$$d(Sx, Ty) \precsim \lambda(x, y, a)d(x, y) + \mu(x, y, a)\tfrac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + \gamma(x, y, a)\tfrac{d(y,Sx)d(x,Ty)}{1+d(x,y)}$$
$$+\delta(x, y, a)\tfrac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{1+d(x,Ty)+d(y,Sx)},$$

(c) $\lambda(x, y, a) + \mu(x, y, a) + \gamma(x, y, a) + \delta(x, y, a) < 1$, then S and T have a unique common fixed point.

In this chapter we generalize the Theorem 4.1 in complex valued S-metric spaces for four maps satisfying more general contractive condition using 7 functions.

First we prove a proposition which is needed to prove our main Theorem.

Proposition 4.2: Let (X, S) be a complex valued S-metric space and $F, G, f, g : X \to X$. Let $y_0 \in X$ and define the sequence $\{y_n\}$ by

$y_{2n+1} = gx_{2n+1} = Fx_{2n}; \quad y_{2n+2} = fx_{2n+2} = Gx_{2n+1}. \; \forall \; n = 0, 1, 2....$

Assume that there exists a mapping $\lambda_1 : X \times X \times X \to \mathbb{R}^+$ such that

(i) $\lambda_1(Fx, y, a) \leq \lambda_1(fx, y, a)$ and $\lambda_1(x, Gy, a) \leq \lambda_1(x, gy, a)$,

(ii) $\lambda_1(Gx, y, a) \leq \lambda_1(gx, y, a)$ and $\lambda_1(x, Fy, a) \leq \lambda_1(x, fy, a)$.

$\forall \; x, y \in X$ and for a fixed element $a \in X$ and $n = 0, 1, 2, ...$

Then $\lambda_1(y_{2n}, y, a) \leq \lambda_1(y_0, y, a)$ and

$\lambda_1(x, y_{2n+1}, a) \leq \lambda_1(x, y_1, a), \forall \; x, y \in X.$

Proof: Let $x, y \in X$ and $n = 0, 1, 2...$ Then we have

$\lambda_1(y_{2n}, y, a) = \lambda_1(Gx_{2n-1}, y, a) \leq \lambda_1(gx_{2n-1}, y, a)$

$= \lambda_1(y_{2n-1}, y, a) = \lambda_1(Fx_{2n-2}, y, a) \leq \lambda_1(fx_{2n-2}, y, a)$

$= \lambda_1(y_{2n-2}, y, a) = \lambda_1(Gx_{2n-3}, y, a) \leq \lambda_1(gx_{2n-3}, y, a)$

$= \lambda_1(y_{2n-3}, y, a) \cdots = \lambda_1(y_0, y, a).$

Thus $\lambda_1(y_{2n}, y, a) \leq \lambda_1(y_0, y, a).$

Similarly we have

$\lambda_1(x, y_{2n+1}, a) = \lambda_1(x, Fx_{2n}, a) \leq \lambda_1(x, fx_{2n}, a)$

$= \lambda_1(x, y_{2n}, a) = \lambda_1(x, Gx_{2n-1}, a) \leq \lambda_1(x, gx_{2n-1}, a)$

$= \lambda_1(x, y_{2n-1}, a) = \lambda_1(x, Fx_{2n-2}, a) \leq \lambda_1(x, fx_{2n-2}, a)$

$= \lambda_1(x, y_{2n-2}, a) \cdots = \lambda_1(x, y_1, a).$

Thus $\lambda_1(x, y_{2n+1}, a) \leq \lambda_1(x, y_1, a).$

Now we give our main theorem.

Theorem 4.3. Let (X, S) be a complex valued S-metric space and $F, G, f, g : X \to X$ satisfying the conditions .

(4.3.1) $GX \subseteq fX$ and $FX \subseteq gX$,

(4.3.2) The pairs (F, f) and (G, g) are weakly compatible ,

(4.3.3) fX or gX is a complete subspace of X,

(4.3.4) If there exist mappings $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 : X \times X \times X \to \mathbb{R}^+$ such that $\lambda_n(Fx, y, a) \leq \lambda_n(fx, y, a)$; $\lambda_n(Gx, y, a) \leq \lambda_n(gx, y, a)$ and $\lambda_n(x, Fy, a) \leq \lambda_n(x, fy, a)$; $\lambda_n(x, Gy, a) \leq \lambda_n(x, gy, a), \forall\, n = 1, 2, 3..., 7$, forall $x,\ y\ \in\ X$ and for a fixed element $a \in X$,

(4.3.5)

$$\begin{aligned} S(Fx, Fx, Gy) \precsim\ & \lambda_1(fx, gy, a)S(fx, fx, gy) + \lambda_2(fx, gy, a)S(fx, fx, Fx) \\ & + \lambda_3(fx, gy, a)S(gy, gy, Gy) \\ & + \lambda_4(fx, gy, a)[S(gy, gy, Fx) + S(fx, fx, Gy)] \\ & + \lambda_5(fx, gy, a)\left(\tfrac{S(fx,fx,Fx)S(gy,gy,Gy)}{1+S(fx,fx,gy)}\right) \\ & + \lambda_6(fx, gy, a)\left(\tfrac{S(gy,gy,Fx)S(fx,fx,Gy)}{1+S(fx,fx,gy)}\right) \\ & + \lambda_7(fx, gy, a)\left(\tfrac{S(fx,fx,Fx)S(fx,fx,Gy)+S(gy,gy,Gy)S(gy,gy,Fx)}{1+S(fx,fx,Gy)+S(gy,gy,Fx)}\right) \end{aligned}$$

$\forall\ x,\ y\ \in\ X$ and for a fixed element $a\ \in\ X$, where

(4.3.6) $(\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + \lambda_7)(x, y, a) < 1$.

Then F, G, f and g have a unique common fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point.

We define a sequence $\{y_n\}$ in X such that $y_{2n+1} = gx_{2n+1} = Fx_{2n}$ and $y_{2n+2} = fx_{2n+2} = Gx_{2n+1}, n = 0, 1, 2, ...$

From (4.3.5), we have

$$
\begin{aligned}
& S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \\
& = S(Fx_{2n}, Fx_{2n}, Gx_{2n+1}) \\
& \precsim \lambda_1(y_{2n}, y_{2n+1}, a) S(y_{2n}, y_{2n}, y_{2n+1}) + \lambda_2(y_{2n}, y_{2n+1}, a) S(y_{2n}, y_{2n}, y_{2n+1}) \\
& \quad + \lambda_3(y_{2n}, y_{2n+1}, a) S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \\
& \quad + \lambda_4(y_{2n}, y_{2n+1}, a) [S(y_{2n+1}, y_{2n+1}, y_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+2})] \\
& \quad + \lambda_5(y_{2n}, y_{2n+1}, a) \left(\tfrac{S(y_{2n}, y_{2n}, y_{2n+1}) S(y_{2n+1}, y_{2n+1}, y_{2n+2})}{1 + S(y_{2n}, y_{2n}, y_{2n+1})} \right) \\
& \quad + \lambda_6(y_{2n}, y_{2n+1}, a) \left(\tfrac{S(y_{2n+1}, y_{2n+1}, y_{2n+1}) S(y_{2n}, y_{2n}, y_{2n+1})}{1 + S(y_{2n}, y_{2n}, y_{2n+1})} \right) \\
& \quad + \lambda_7(y_{2n}, y_{2n+1}, a) \left(\tfrac{S(y_{2n}, y_{2n}, y_{2n+1}) S(y_{2n}, y_{2n}, y_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2}) S(y_{2n+1}, y_{2n+1}, y_{2n+1})}{1 + S(y_{2n}, y_{2n}, y_{2n+1}) + S(y_{2n+1}, y_{2n+1}, y_{2n+1})} \right)
\end{aligned}
$$

Since $S(x, x, x) = 0$, we have

$$
\begin{aligned}
|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leq {} & \lambda_1(y_{2n}, y_{2n+1}, a) \left| S(y_{2n}, y_{2n}, y_{2n+1}) \right| \\
& + \lambda_2(y_{2n}, y_{2n+1}, a) \left| S(y_{2n}, y_{2n}, y_{2n+1}) \right| \\
& + \lambda_3(y_{2n}, y_{2n+1}, a) \left| S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \right| \\
& + \lambda_4(y_{2n}, y_{2n+1}, a) \left| S(y_{2n}, y_{2n}, y_{2n+1}) \right| \\
& + \lambda_4(y_{2n}, y_{2n+1}, a) \left| S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \right| \\
& + \lambda_5(y_{2n}, y_{2n+1}, a) \left| S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \right| \left| \tfrac{S(y_{2n}, y_{2n}, y_{2n+1})}{1 + S(y_{2n}, y_{2n}, y_{2n+1})} \right| \\
& + \lambda_7(y_{2n}, y_{2n+1}, a) \left| S(y_{2n}, y_{2n}, y_{2n+1}) \right| \left| \tfrac{S(y_{2n}, y_{2n}, y_{2n+2})}{1 + S(y_{2n}, y_{2n}, y_{2n+2})} \right|.
\end{aligned}
$$

$$
\begin{aligned}
|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leq {} & (\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7)(y_{2n}, y_{2n+1}, a) \left| S(y_{2n}, y_{2n}, y_{2n+1}) \right| \\
& + (\lambda_3 + \lambda_4 + \lambda_5)(y_{2n}, y_{2n+1}, a) \left| S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \right|.
\end{aligned}
$$

Using Proposition 4.2, we get

$$
\begin{aligned}
|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leq {} & (\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7)(y_0, y_1, a) \left| S(y_{2n}, y_{2n}, y_{2n+1}) \right| \\
& + (\lambda_3 + \lambda_4 + \lambda_5)(y_0, y_1, a) \left| S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \right|
\end{aligned}
$$

which in turn implies that

$$
|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leq \left(\tfrac{(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7)(y_0, y_1, a)}{1 - (\lambda_3 + \lambda_4 + \lambda_5)(y_0, y_1, a)} \right) \left| S(y_{2n}, y_{2n}, y_{2n+1}) \right|.
$$

Let $h_1 = \left(\frac{(\lambda_1+\lambda_2+\lambda_4+\lambda_7)(y_0,y_1,a)}{1-(\lambda_3+\lambda_4+\lambda_5)(y_0,y_1,a)}\right)$

Thus $|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leq h_1 |S(y_{2n}, y_{2n}, y_{2n+1})|$. (1)

Similarly using $S(x, y, y) = S(x, x, y)$ and proceeding as above we can show that $|S(y_{2n+2}, y_{2n+2}, y_{2n+3})| \leq h_2 |S(y_{2n+1}, y_{2n+1}, y_{2n+2})|$ (2)

where $h_2 = \left(\frac{(\lambda_1+\lambda_3+\lambda_4+\lambda_7)(y_0,y_1,a)}{1-(\lambda_2+\lambda_4+\lambda_5)(y_0,y_1,a)}\right)$.

Let $h = max\{h_1, h_2\}$, then $0 \leq h < 1$, since $h_1, h_2 \in \mathbb{R}^+$.

Hence from (1) and (2), we have $|S(y_n, y_n, y_{n+1})| \leq h |S(y_{n-1}, y_{n-1}, y_n)|$, $n = 1, 2, ...$

Hence

$$\begin{aligned}
|S(y_k, y_k, y_{k+1})| &\leq h |S(y_{k-1}, y_{k-1}, y_k)| \\
&\leq h^2 |S(y_{k-2}, y_{k-2}, y_{k-1})| \\
&\vdots \\
&\vdots \\
&\leq h^k |S(y_0, y_0, y_1)| \quad (3) \\
&\to 0 \quad as \quad k \to \infty \quad (4)
\end{aligned}$$

Hence for any $m > n$, we have

$$\begin{aligned}
|S(y_n, y_n, y_m)| &= 2 \left[\begin{array}{l} |S(y_n, y_n, y_{n+1})| + |S(y_{n+1}, y_{n+1}, y_{n+2})| + \\ ...+ |S(y_{m-1}, y_{m-1}, y_m)| \end{array} \right] \\
&= 2(h^n + h^{n+1} + ... + h^{m-1}) |S(y_0, y_0, y_1)| \, from(3) \\
&\leq \tfrac{2h^n}{1-h} |S(y_0, y_0, y_1)|
\end{aligned}$$

$|S(y_n, y_n, y_m)| \leq \frac{2h^n}{1-h} |S(y_0, y_0, y_1)| \to 0$ as $m, n \to \infty$.

Hence $\{y_n\}$ is a Cauchy sequence in X.

Now suppose fX is a complete subspace of X. Since $y_{2n+2} = fx_{2n+2} \in f(X)$ and $\{y_n\}$ is a Cauchy sequence, there exists $z \in f(X)$ such that $y_{2n+2} \to z$ as $n \to \infty$.

Then there exists $u \in X$ such that $fu = z$.

Thus $\lim_{n\to\infty} Fx_{2n} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Gx_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = z$.

Consider

$$\begin{aligned}
&S(Fu, Fu, Gx_{2n+1})\\
&\quad\precsim \lambda_1(fu, y_{2n+1}, a)S(fu, fu, y_{2n+1})\\
&\qquad +\lambda_2(fu, y_{2n+1}, a)S(fu, fu, Fu)\\
&\qquad +\lambda_3(fu, y_{2n+1}, a)S(y_{2n+1}, y_{2n+1}, y_{2n+2})\\
&\qquad +\lambda_4(fu, y_{2n+1}, a)[S(y_{2n+1}, y_{2n+1}, Fu) + S(fu, fu, y_{2n+2})]\\
&\qquad +\lambda_5(fu, y_{2n+1}, a)\left(\tfrac{S(fu,fu,Fu)S(y_{2n+1},y_{2n+1},y_{2n+2})}{1+S(fu,fu,y_{2n+1})}\right)\\
&\qquad +\lambda_6(fu, y_{2n+1}, a)\left(\tfrac{S(y_{2n+1},y_{2n+1},Fu)S(fu,fu,y_{2n+2})}{1+S(fu,fu,y_{2n+1})}\right)\\
&\qquad +\lambda_7(fu, y_{2n+1}, a)\left(\tfrac{S(fu,fu,Fu)S(fu,fu,y_{2n+2})+S(y_{2n+1},y_{2n+1},y_{2n+2})S(y_{2n+1},y_{2n+1},Fu)}{1+S(fu,fu,y_{2n+2})+S(y_{2n+1},y_{2n+1},Fu)}\right)
\end{aligned}$$

$$\begin{aligned}
&|S(Fu, Fu, Gx_{2n+1})|\\
&\quad\le \lambda_1(fu, y_{2n+1}, a)|S(fu, fu, y_{2n+1})|\\
&\qquad +\lambda_2(fu, y_{2n+1}, a)|S(fu, fu, Fu)|\\
&\qquad +\lambda_3(fu, y_{2n+1}, a)|S(y_{2n+1}, y_{2n+1}, y_{2n+2})|\\
&\qquad +\lambda_4(fu, y_{2n+1}, a)|S(y_{2n+1}, y_{2n+1}, Fu) + S(fu, fu, y_{2n+2})|\\
&\qquad +\lambda_5(fu, y_{2n+1}, a)\left(\tfrac{|S(fu,fu,Fu)||S(y_{2n+1},y_{2n+1},y_{2n+2})|}{|1+S(fu,fu,y_{2n+1})|}\right)\\
&\qquad +\lambda_6(fu, y_{2n+1}, a)\left(\tfrac{|S(y_{2n+1},y_{2n+1},Fu)||S(fu,fu,y_{2n+2})|}{|1+S(fu,fu,y_{2n+1})|}\right)\\
&\qquad +\lambda_7(fu, y_{2n+1}, a)\left(\tfrac{|S(fu,fu,Fu)||S(fu,fu,y_{2n+2})|+|S(y_{2n+1},y_{2n+1},y_{2n+2})||S(y_{2n+1},y_{2n+1},Fu)|}{|1+S(fu,fu,y_{2n+2})+S(y_{2n+1},y_{2n+1},Fu)|}\right).
\end{aligned}$$

Letting $n \longrightarrow \infty$ and using Lemma 1.8.5 (Ch-1) and Lemma 1.8.7 (Ch-1), we get

$|S(Fu, Fu, z)| \le \lambda_2(z, z, a)\,|S(z, z, Fu)| + \lambda_4(z, z, a)\,|S(z, z, Fu)|$

from (4), Lemma 1.8.5 (Ch-1)

$(1 - (\lambda_2 + \lambda_4)(z, z, a))\,|S(z, z, Fu)| \le 0$

which in turn yields from (4.3.6) that $|S(Fu, Fu, z)| \le 0$.

Therefore $|S(Fu, Fu, z)| = 0$. Hence $Fu = z$. Thus $fu = Fu = z$.

Since $FX \subseteq gX$, there exists $v \in X$ such that $Fu = gv$.

Thus $fu = Fu = gv = z$.

Again from (4.3.5), we have

$$\begin{aligned}
|S(z, z, Gv)| = |S(Fu, Fu, Gv)| \\
&\leq \lambda_1(fu, gv, a)|S(fu, fu, gv)| + \lambda_2(fu, gv, a)|S(fu, fu, Fu)| \\
&+\lambda_3(fu, gv, a)|S(gv, gv, Gv)| \\
&+\lambda_4(fu, gv, a)|S(gv, gv, Fu) + S(fu, fu, Gv)| \\
&+\lambda_5(fu, gv, a)\left(\frac{|S(fu,fu,Fu)||S(gv,gv,Gv)|}{|1+S(fu,fu,gv)|}\right) \\
&+\lambda_6(fu, gv, a)\left(\frac{|S(gv,gv,Fu)||S(fu,fu,Gv)|}{|1+S(fu,fu,gv)|}\right) \\
&+\lambda_7(fu, gv, a)\left(\frac{|S(fu,fu,Fu)||S(fu,fu,Gv)|+|S(gv,gv,Gv)||S(gv,gv,Fu)|}{|1+S(fu,fu,Gv)+S(gv,gv,Fu)|}\right)
\end{aligned}$$

so that

$|S(z, z, Gv)| \leq \lambda_3(z, z, a)\,|S(z, z, Gv)| + \lambda_4(z, z, a)\,|S(z, z, Gv)|$.

$(1 - (\lambda_3 + \lambda_4)(z, z, a))\,|S(z, z, Gv)| \leq 0$

which in turn yields from (4.3.6) that $|S(z, z, Gv)| \leq 0$.

Therefore $|S(z, z, Gv)| = 0$. Hence $Gv = z$.

Thus $Gv = z = fu = Fu = gv$. (5)

Since (F, f) is weakly compatible,

we have $fz = fFu = Ffu = Fz$. (6)

$$\begin{aligned}
S(Fz, Fz, z) = S(Fz, Fz, Gv) \\
&\precsim \lambda_1(fz, gv, a)S(fz, fz, gv) + \lambda_2(fz, gv, a)S(fz, fz, Fz) \\
&+\lambda_3(fz, gv, a)S(gv, gv, Gv) \\
&+\lambda_4(fz, gv, a)[S(gv, gv, Fz) + S(fz, fz, Gv)] \\
&+\lambda_5(fz, gv, a)\left(\frac{S(fz,fz,Fz)S(gv,gv,Gv)}{1+S(fz,fz,Gv)}\right)
\end{aligned}$$

$$+\lambda_6(fz,gv,a)\left(\frac{S(gv,gv,Fz)S(fz,fz,Gv)}{1+S(fz,fz,Gv)}\right)$$
$$+\lambda_7(fz,gv,a)\left(\frac{S(fz,fz,Fz)S(fz,fz,Gv)+S(gv,gv,Gv)S(gv,gv,Fz)}{1+S(fz,fz,Gv)+S(gv,gv,Fz)}\right)$$
$$=\lambda_1(Fz,z,a)S(Fz,Fz,z)$$
$$+\lambda_4(Fz,z,a)[S(z,z,Fz)+S(Fz,Fz,z)]$$
$$+\lambda_6(Fz,z,a)\left(\frac{S(z,z,Fz)S(Fz,Fz,z)}{1+S(Fz,Fz,z)}\right) \textit{ from } (5) \textit{ and } (6)$$

$$|S(Fz,Fz,z)| \leq \lambda_1(Fz,z,a)\,|S(Fz,Fz,z)| + \lambda_4(Fz,z,a)\,|S(z,z,Fz)+S(Fz,Fz,z)|$$
$$+\lambda_6(Fz,z,a)\,|S(z,z,Fz)|\left|\frac{S(Fz,Fz,z)}{1+S(Fz,Fz,z)}\right|.$$

$$(1-(\lambda_1+2\lambda_4+\lambda_6)(Fz,z,a))\,|S(Fz,Fz,z)| \leq 0$$

which in turn yields from (4.3.6) that $|S(Fz,Fz,z)| \leq 0$.

Therefore $|S(Fz,Fz,z)| = 0$. Hence $Fz = z$.Thus $z = Fz = fz$. (7)

Since the pair (G,g) is weakly compatible,

we have $gz = gGv = Ggv = Gz$. (8)

From (4.3.5)

$$S(z,z,Gz) = S(Fz,Fz,Gz)$$
$$\precsim \lambda_1(fz,gz,a)S(fz,fz,gz) + \lambda_2(fz,gz,a)S(fz,fz,Fz)$$
$$+\lambda_3(fz,gz,a)S(gz,gz,Gz) + \lambda_4(fz,gz,a)[S(gz,gz,Fz)+S(fz,fz,Gz)]$$
$$+\lambda_5(fz,gz,a)\left(\frac{S(fz,fz,FZ)S(gz,gz,Gz)}{1+S(fz,fz,gz)}\right)$$
$$+\lambda_6(fz,gz,a)\left(\frac{S(gz,gz.Fz)S(fz,fz,Gz)}{1+S(fz,fz,gz)}\right)$$
$$+\lambda_7(fz,gz,a)\left(\frac{S(fz,fz,Fz)S(fz,fz,Gz)+S(gz,gz,Gz)S(gz,gz,Fz)}{1+S(fz,fz,Gz)+S(gz,gz,Fz)}\right)$$

$$|S(z,z,Gz)| \leq \lambda_1(z,Gz,a)\,|S(z,z,Gz)| + \lambda_4(z,Gz,a)\,|S(Gz,Gz,z)+S(z,z,Gz)|$$
$$+\lambda_6(z,Gz,a)\,|S(Gz,Gz,z)|\left|\frac{S(z,z,Gz)}{1+S(z,z,Gz)}\right| \textit{ from } (7),(8)$$

$$(1-(\lambda_1+2\lambda_4+\lambda_6)(z,Gz,a))\,|S(z,z,Gz)| \leq 0$$

which in turn yields from (4.3.6) that $|S(z,z,Gz)| \leq 0$.

Therefore $|S(z,z,Gz)| = 0$.Hence $Gz = z$, so that $Gz = gz = z$. (9)

Thus from (7) and (9), z is a common fixed point of F, G, f and g.

For uniqueness, let $z^* \in X$ be such that $fz^* = Fz^* = z^* = gz^* = Gz^*$.

Now from (4.3.5)

$$\begin{aligned}
S(z, z, z^*) = S(Fz, Fz, Gz^*) \\
&\precsim \lambda_1(fz, gz^*, a)S(fz, fz, gz^*) + \lambda_2(fz, gz^*, a)S(fz, fz, Fz) \\
&+\lambda_3(fz, gz^*, a)S(gz^*, gz^*, Gz^*) \\
&+\lambda_4[(fz, gz^*, a)[S(gz^*, gz^*, Fz) + S(fz, fz, Gz^*)] \\
&+\lambda_5(fz, gz^*, a)\left(\tfrac{S(fz,fz,Fz)S(gz^*,gz^*,Gz^*)}{1+S(fz,fz,gz^*)}\right) \\
&+\lambda_6(fz, gz^*, a)\left(\tfrac{S(gz^*,gz^*,Fz)S(fz,fz,Gz^*)}{1+S(fz,fz,gz^*)}\right) \\
&+\lambda_7(fz, gz^*, a)\left(\tfrac{S(fz,fz,Fz)S(fz,fz,Gz^*)+S(gz^*,gz^*,Gz^*)S(gz^*,gz^*,Fz)}{1+S(fz,fz,Gz^*)+S(gz^*,gz^*,Fz)}\right).
\end{aligned}$$

$$\begin{aligned}
|S(z, z, z^*)| \leq \lambda_1(z, z^*, a)\,|S(z, z, z^*)| \\
&+\lambda_4(z, z^*, a)\,|S(z^*, z^*, z) + S(z, z, z^*)| \\
&+\lambda_6(z, z^*, a)\,|S(z^*, z^*, z)| \left|\tfrac{S(z,z,z^*)}{1+S(z,z,z^*)}\right|.
\end{aligned}$$

$|S(z, z, z^*)| \leq (\lambda_1 + 2\lambda_4 + \lambda_6)(z, z^*, a)\,|S(z, z, z^*)|$.

$(1 - (\lambda_1 + 2\lambda_4 + \lambda_6)(z, z^*, a))\,|S(z, z, z^*)| \leq 0$

which in turn yields from (4.3.6) that $|S(z, z, z^*)| \leq 0$.

Therfore $|S(z, z, z^*)| = 0$. Thus $z = z^*$.

Hence z is the unique common fixed point of F, G, f and g.

Similarly we can prove the theorem if gX is a complete subspace of X.

Now we give an example to illustrate our main Theorem 4.3.

Example 4.4. Let $X = [0, 1]$ and $S : X \times X \times X \to C$ be defined by $S(x, y, z) = |x-z|+i|y-z|$. Then (X, S) is a complex valued S- metric space. Define F, G, f and $g : X \to X$ by $Fx = \frac{x}{16}, Gx = \frac{x}{12}, fx = \frac{x}{4}$ and $gx = \frac{x}{3}$, for all $x \in X$. For fixed element $a = \frac{1}{3}$,

define $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 : X \times X \times X \to [0, 1]$ by

$\lambda_1(x,y,a) = (\frac{x}{40} + \frac{y}{50} + a), \lambda_2(x,y,a) = \frac{xya}{10}, \lambda_3(x,y,a) = \frac{x^2y^2a^2}{10},$

$\lambda_4(x,y,a) = \frac{x^3y^3a^3}{10}, \lambda_5(x,y,a) = \frac{x^3+y^3+a^3}{10}, \lambda_6(x,y,a) = \frac{x^2ya^3}{50},$

$\lambda_7(x,y,a) = \frac{xy^3a^2}{40}, \forall\ x,\ y\ \in\ X.$

Then

$\lambda_1(x,y,a) + \lambda_2(x,y,a) + \lambda_3(x,y,a) + 2\lambda_4(x,y,a) + \lambda_5(x,y,a)$

$+ \lambda_6(x,y,a) + \lambda_7(x,y,a)$

$= (\frac{x}{40} + \frac{y}{50} + a) + \frac{xya}{10} + \frac{x^2y^2a^2}{10} + 2(\frac{x^3y^3a^3}{10}) + \frac{x^3+y^3+a^3}{10} + \frac{x^2ya^3}{50} + \frac{xy^3a^2}{40}$

$\leq (\frac{1}{40} + \frac{1}{50} + \frac{1}{3}) + \frac{1}{30} + \frac{1}{90} + \frac{2}{270} + \frac{55}{270} + \frac{1}{1350} + \frac{1}{360}$

$= \frac{3442}{5400} < 1.$

Hence $(\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + \lambda_7)(x,y,a) < 1.$

We have $\lambda_1(Fx,y,a) = \lambda_1(\frac{x}{16},y,a) = (\frac{x}{640} + \frac{y}{50} + a)$ and

$\lambda_1(fx,y,a) = \lambda_1(\frac{x}{4},y,a) = (\frac{x}{160} + \frac{y}{50} + a)$

clearly $\lambda_1(Fx,y,a) \leq \lambda_1(fx,y,a).$

We have$\lambda_1(x,Fy,a) = \lambda_1(x,\frac{y}{16},a) = (\frac{x}{40} + \frac{y}{800} + a)$ and

$\lambda_1(x,fy,a) = \lambda_1(x,\frac{y}{4},a) = (\frac{x}{40} + \frac{y}{200} + a)$

clearly$\lambda_1(x,Fy,a) \leq \lambda_1(x,fy,a).$

We have $\lambda_1(Gx,y,a) = \lambda_1(\frac{x}{12},y,a) = (\frac{x}{480} + \frac{y}{50} + a)$ and

$\lambda_1(gx,y,a) = \lambda_1(\frac{x}{3},y,a) = (\frac{x}{120} + \frac{y}{50} + a)$

clearly $\lambda_1(Gx,y,a) \leq \lambda_1(gx,y,a).$

We have $\lambda_1(x,Gy,a) = \lambda_1(x,\frac{y}{12},a) = (\frac{x}{40} + \frac{y}{600} + a)$ and

$\lambda_1(x,gy,a) = \lambda_1(x,\frac{y}{3},a) = (\frac{x}{40} + \frac{y}{150} + a)$

clearly $\lambda_1(x,Gy,a) \leq \lambda_1(x,gy,a).$

Similarly we can prove that

$\lambda_n(Fx,y,a) \leq \lambda_n(fx,y,a), \lambda_n(x,Fy,a) \leq \lambda_n(x,fy,a)$ and

$\lambda_n(Gx,y,a) \leq \lambda_n(gx,y,a), \lambda_n(x,Gy,a) \leq \lambda_n(x,gy,a)\ \forall\ n\ =\ 2,3,4,...7.$

Consider

$$
\begin{aligned}
&|S(Fx, Fx, Gy)| \\
&= \left|S(\tfrac{x}{16}, \tfrac{x}{16}, \tfrac{y}{12})\right| \\
&= |\tfrac{x}{16} - \tfrac{y}{12}| + i|\tfrac{x}{16} - \tfrac{y}{12}| \\
&= \tfrac{1}{4}[|\tfrac{x}{4} - \tfrac{y}{3}| + i|\tfrac{x}{4} - \tfrac{y}{3}|] \\
&< \tfrac{1}{3}[|\tfrac{x}{4} - \tfrac{y}{3}| + i|\tfrac{x}{4} - \tfrac{y}{3}|] \\
&\leq (\tfrac{x}{160} + \tfrac{y}{150} + \tfrac{1}{3})[|\tfrac{x}{4} - \tfrac{y}{3}| + i|\tfrac{x}{4} - \tfrac{y}{3}|] \\
&= \lambda_1(fx, gy, a)S(fx, fx, gy) \\
&\leq \lambda_1(fx, gy, a)S(fx, fx, gy) + \lambda_2(fx, gy, a)S(fx, fx, Fx) \\
&\quad +\lambda_3(fx, gy, a)S(gy, gy, Gy) + \lambda_4(fx, gy, a)[S(gy, gy, Fx) + S(fx, fx, Gy)] \\
&\quad +\lambda_5(fx, gy, a)\left(\tfrac{S(fx,fx,Fx)S(gy,gy,Gy)}{1+S(fx,fx,gy)}\right) + \lambda_6(fx, gy, a)\left(\tfrac{S(gy,gy,Fx)S(fx,fx,Gy)}{1+S(fx,fx,gy)}\right) \\
&\quad +\lambda_7(fx, gy, a)\left(\tfrac{S(fx,fx,Fx)S(fx,fx,Gy)+S(gy,gy,Gy)S(gy,gy,Fx)}{1+S(fx,fx,Gy)+S(gy,gy,Fx)}\right).
\end{aligned}
$$

Thus (4.3.5) is satisfied.

One can easily verify the remaining conditions of Theorem 4.3.

Clearly $x = 0$ is the unique common fixed point of F, G, f and g.

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CHAPTER 5

COMMON AND COUPLED FIXED POINT THEOREMS IN S_b-METRIC SPACES

We divide Chapter 5 into two sections, namely, Section 5.1 and Section 5.2. The main aim of this Chapter is to prove common fixed point results and coupled Suzuki type result in complex valued S_b and S_b metric spaces.

SECTION 5.1: UNIQUE COMMON FIXED POINT THEOREM FOR FOUR MAPS IN COMPLEX VALUED S_b-METRIC SPACES

Recently N.Priyobarta et al.[72] proved the following theorem in complex valued S_b-metric spaces as follows.

Theorem 5.1.1.(N.Priyobarta et al.[72]): Let (X, S) be a complete complex valued S_b-metric space and the mapping $f : X \to X$ satisfies for every $x, y \in X$

$$S(fx, fx, fy) \precsim \alpha(S(x, x, fx) + S(y, y, fy))$$

where $\alpha \in [0, \frac{1}{2})$. Then f has a unique fixed point.

In this section we generalize the Theorem 5.1.1 for two weakly compatible pairs of mappings satisfying a contractive condition in complex valued S_b-metric spaces. An example is also given to validate our result.

Theorem 5.1.2. Let (X, S) be a complex valued S_b-metric space with coefficient $b > 1$ and $F, G, f, g : X \to X$ satisfying the conditions.

(5.1.2.1) $GX \subseteq fX$ and $FX \subseteq gX$,

(5.1.2.2) the pairs (F, f) and (G, g) are weakly compatible ,

(5.1.2.3) fX or gX is a complete subspace of X, (5.1.2.4) $S(Fx, Fx, Gy) \precsim \alpha\ max \left\{ \begin{array}{l} S(fx, fx, gy), S(fx, fx, Fx), \\ S(gy, gy, Gy), \frac{S(fx,fx,Fx)S(gy,gy,Gy)}{1+S(Fx,Fx,Gy)} \end{array} \right\}.$

$\forall\ x, y \in X$ and α is real with $0 < \alpha < \frac{1}{2b}$.

Then F, G, f and g have a unique common fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point.

We define a sequence $\{y_n\}$ in X such that $y_{2n+1} = gx_{2n+1} = Fx_{2n}$ and

$y_{2n+2} = fx_{2n+2} = Gx_{2n+1}, n = 0, 1, 2, ...$

From (5.1.2.4), we have

$$\begin{aligned} & S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \\ & = S(Fx_{2n}, Fx_{2n}, Gx_{2n+1}) \\ & \precsim \alpha\ \max \left\{ \begin{array}{l} S(fx_{2n}, fx_{2n}, gx_{2n+1}), S(fx_{2n}, fx_{2n}, Fx_{2n}), \\ S(gx_{2n+1}, gx_{2n+1}, Gx_{2n+1}), \frac{S(fx_{2n},fx_{2n},Fx_{2n})S(gx_{2n+1},gx_{2n+1},Gx_{2n+1})}{1+S(Fx_{2n},Fx_{2n},Gx_{2n+1})} \end{array} \right\} \\ & = \alpha\ \max \left\{ \begin{array}{l} S(y_{2n}, y_{2n}, y_{2n+1}), S(y_{2n}, y_{2n}, y_{2n+1}), \\ S(y_{2n+1}, y_{2n+1}, y_{2n+2}), \frac{S(y_{2n},y_{2n},y_{2n+1})S(y_{2n+1},y_{2n+1},y_{2n+2})}{1+S(y_{2n+1},y_{2n+1},y_{2n+2})} \end{array} \right\} \\ & = \alpha\ \max\ \{S(y_{2n}, y_{2n}, y_{2n+1}), S(y_{2n+1}, y_{2n+1}, y_{2n+2})\}. \end{aligned}$$

If we assume that $S(y_{2n+1}, y_{2n+1}, y_{2n+2}) > S(y_{2n}, y_{2n}, y_{2n+1})$.

Then $S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \precsim \alpha S(y_{2n+1}, y_{2n+1}, y_{2n+2})$

$|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leq \alpha\, |S(y_{2n}, y_{2n}, y_{2n+1})|$

$(1 - \alpha)\, |S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leq 0$

Since $0 < \alpha < 1$, we get $|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leq 0$.

It is a contradiction .

Thus $|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leq \alpha\, |S(y_{2n}, y_{2n}, y_{2n+1})|$. (1)

Now again from (5.1.2.4)

$$S(y_{2n+2}, y_{2n+2}, y_{2n+3})$$
$$= S(Gx_{2n+1}, Gx_{2n+1}, Fx_{2n+2}).\ Since\ S(x, x, y) = S(y, y, x),\ weget$$
$$= S(Fx_{2n+2}, Fx_{2n+2}, Gx_{2n+1})$$
$$\precsim \alpha\ \max \left\{ \begin{array}{l} S(fx_{2n+2}, fx_{2n+2}, gx_{2n+1}), S(fx_{2n+2}, fx_{2n+2}, Fx_{2n+2}), \\ S(gx_{2n+1}, gx_{2n+1}, Gx_{2n+1}), \frac{S(fx_{2n+2},fx_{2n+2},Fx_{2n+2})S(gx_{2n+1},gx_{2n+1},Gx_{2n+1})}{1+S(Fx_{2n+2},Fx_{2n+2}mGx_{2n+1})} \end{array} \right\}$$
$$= \alpha\ \max \left\{ \begin{array}{l} S(y_{2n+2}, y_{2n+2}, y_{2n+1}), S(y_{2n+2}, y_{2n+2}, y_{2n+3}) \\ S(y_{2n+1}, y_{2n+1}, y_{2n+2}), \frac{S(y_{2n+2},y_{2n+2},y_{2n+3})S(y_{2n+1},y_{2n+1},y_{2n+2})}{1+S(y_{2n+3},y_{2n+3},y_{2n+2})} \end{array} \right\}$$
$$= \alpha\ \max\ \{S(y_{2n+1}, y_{2n+1}, y_{2n+2}), S(y_{2n+2}, y_{2n+2}, y_{2n+3})\}.$$

If we assume that $S(y_{2n+2}, y_{2n+2}, y_{2n+3}) > S(y_{2n+1}, y_{2n+1}, y_{2n+2})$.

Then $S(y_{2n+2}, y_{2n+2}, y_{2n+3}) \precsim \alpha\ S(y_{2n+2}, y_{2n+2}, y_{2n+3})$

$|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leq \alpha\,|S(y_{2n}, y_{2n}, y_{2n+1})|$

$(1-\alpha)\,|S(y_{2n+2}, y_{2n+2}, y_{2n+3})| \leq 0.$

Since $0 < \alpha < 1$, we have $|S(y_{2n+2}, y_{2n+2}, y_{2n+3})| \leq 0$.

It is a contradiction .

Thus $|S(y_{2n+2}, y_{2n+2}, y_{2n+3})| \leq \alpha\,|S(y_{2n+1}, y_{2n+1}, y_{2n+2})|$. (2)

Continuing in this way , we get

$$|S(y_n, y_n, y_{n+1})| \leq \alpha\,|S(y_{n-1}, y_{n-1}, y_n)|\,,\ for\ n = 1, 2, 3, ...$$

Hence

$$\begin{aligned} |S(y_k, y_k, y_{k+1}| &\leq \alpha\,|S(y_{k-1}, y_{k-1}, y_k| \\ &\leq \alpha^2\,|S(y_{k-2}, y_{k-2}, y_{k-1}| \\ &: \\ &: \\ &\leq \alpha^k\,|S(y_0, y_0, y_1| \qquad (3) \\ &\longrightarrow\ 0\ as\ k\ \longrightarrow\ \infty \end{aligned}$$

Hence foe any $m > n$ we have

$$
\begin{aligned}
&|S(y_n, y_n, y_m)| \\
&\leq 2b \begin{bmatrix} |S(y_n, y_n, y_{n+1})| + b\,|S(y_{n+1}, y_{n+1}, y_{n+2})| + b^2\,|S(y_{n+2}, y_{n+2}, y_{n+3})| + \\ \ldots\ldots\ldots\ldots + b^{m-n-1}\,|S(y_{m-1}, y_{m-1}, y_m)| \end{bmatrix} \\
&\leq 2b \begin{bmatrix} \alpha^n\,|S(y_0, y_0, y_1)| + b\alpha^{n+1}\,|S(y_0, y_0, y_1)| + b^2\alpha^{n+2}\,|S(y_0, y_0, y_1)| + \\ \ldots\ldots\ldots\ldots\ldots + b^{m-n-1}\alpha^{m-1}\,|S(y_0, y_0, y_1)| \end{bmatrix} \\
&\leq 2b\ \alpha^n\,[1 + b\ \alpha + (b\ \alpha)^2 + \ldots\ldots\ldots + (b\ \alpha)^{m-n-1}]\,|S(y_0, y_0, y_1)| \\
&\leq \tfrac{2b\ \alpha^n}{1-b\ \alpha}\,|S(y_0, y_0, y_1)|
\end{aligned}
$$

$|S(y_n, y_n, y_m)| \leq \frac{2b\alpha^n}{1-b\alpha}\,|S(y_0, y_0, y_1)| \to 0$ as $m, n \to \infty$.

Definition 1.10.1(Ch-1), we have

$S(x, y, z) \precsim b(S(x, x, a) + S(y, y, a) + S(z, z, a))$ for all $x, y, z, a \in X$.

By using above condition, we have,

$S(y_n, y_m, y_l) \precsim b(S(y_n, y_n, y_m) + S(y_m, y_m, y_m) + S(y_l, y_l, y_m))$

Letting $n, m, l \to \infty$.

We obtain $|S(y_n, y_m, y_l)| \to 0$.

Thus $\{y_n\}$ is Complex valued S_b-Cauchy sequence.

Now suppose fX is a complete subspace of X. Since $y_{2n+2} = fx_{2n+2} \in f(X)$ and $\{y_n\}$ is a S_b- Cauchy sequence, there exists $z \in f(X)$ such that $y_{2n+2} \to z$ as $n \to \infty$. Then there exists $u \in X$ such that $fu = z$.

Thus $\lim\limits_{n\to\infty} Fx_{2n} = \lim\limits_{n\to\infty} gx_{2n+1} = \lim\limits_{n\to\infty} Gx_{2n+1} = \lim\limits_{n\to\infty} fx_{2n+2} = z$.

Now we show that $Fu = fu = z$.

$$
\begin{aligned}
&S(Fu, Fu, z) \\
&\precsim b[2S(Fu, Fu, Gx_{2n+1}) + S(z, z, Gx_{2n+1}] \\
&= 2b\ S(Fu, Fu, Gx_{2n+1}) + bS(z, z, Gx_{2n+1})
\end{aligned}
$$

$$\precsim 2b\ \alpha\ \max \left\{ \begin{array}{l} S(fu,fu,gx_{2n+1}), S(fu,fu,Fu), \\ S(gx_{2n+1},gx_{2n+1},Gx_{2n+1}), \\ \frac{S(fu,fu,Fu)S(gx_{2n+1},gx_{2n+1},Gx_{2n+1})}{1+S(Fu,Fu,Gx_{2n+1})} \end{array} \right\} + bS(z,z,Gx_{2n+1})$$

letting $n \to \infty$, we get

$$|S(Fu,Fu,z)| \leq 2b\ \alpha\ \max \left\{ \begin{array}{l} |S(z,z,z)|, |S(z,z,Fu)|, \\ |S(z,z,z)|, \frac{|S(z,z,Fu)||S(z,z,z)|}{|1+S(Fu,Fu,z)|} \end{array} \right\} + b|S(z,z,z)|$$

$$= 2b\ \alpha\ |S(Fu,Fu,z)|$$

$$(1-2b\ \alpha)|S(Fu,Fu,z)| \leq 0.$$

Since $0 < \alpha < \frac{1}{2b}$, we get $|S(Fu,z,z)| \leq 0$.

Thus $|(S(Fu,z,z)| = 0$.

Hence $Fu = z$. Thus $fu = Fu = z$.

Since $FX \subseteq gX$, there exists $v \in X$ such that $Fu = gv$.

Thus $fu = Fu = gv = z$.

Now we prove that $Gv = gv = z$.

$$S(z,z,Gv)$$

$$= S(Gv,Gv,z)$$

$$\precsim b\ [2S(Gv,Gv,Fx_{2n}) + S(z,z,Fx_{2n})]$$

$$= 2bS(Gv,Gv,Fx_{2n}) + bS(z,z,Fx_{2n})$$

$$= 2bS(Fx_{2n},Fx_{2n},Gv) + bS(z,z,Fx_{2n})$$

$$\precsim 2b\ \alpha\ \max \left\{ \begin{array}{l} S(fx_{2n},fx_{2n},gv), S(fx_{2n},fx_{2n},Fx_{2n}), \\ S(gv,gv,Gv), \frac{S(fx_{2n},fx_{2n},Fx_{2n})S(gv,gv,Gv)}{|1+S(Fx_{2n},Fx_{2n},Gv)} \end{array} \right\} + bS(z,z,Fx_{2n})$$

letting $n \to \infty$

$$S(z,z,Gv) = 2b\ \alpha\ \max\left\{\begin{array}{l} S(z,z,z), S(z,z,z), \\ S(z,z,Gv), \frac{S(z,z,z)S(z,z,Gv)}{1+S(z,z,Gv)} \end{array}\right\} + bS(z,z,z)$$

$$= 2b\ \alpha\ S(z,z,Gv)$$

$$|S(z,z,Gv)| \leq 2b\ |\alpha S(z,z,Gv)|$$

$$(1-2b\ \alpha)|S(z,z,Gv)| \leq 0.$$

Since $0 < \alpha < \frac{1}{2b}$ such that $|S(z,z,Gv)| \leq 0$. It implies that $|S(z,z,Gv)| = 0$.

Thus $Gv = z$. Hence $Gv = z = fu = Fu = gv$. (4)

Since (F,f) is weakly compatible, we have

$fz = fFu = Ffu = Fz$. (5)

Now

$$S(Fz,Fz,z) \precsim 2bS(Fz,Fz,Gx_{2n+1}) + bS(z,z,Gx_{2n+1})$$

$$\precsim\ 2b\ \alpha\ \max\left\{\begin{array}{l} S(fz,fz,gx_{2n+1}), S(fz,fz,Fz), \\ S(gx_{2n+1},gx_{2n+1},Gx_{2n+1}), \\ \frac{S(fz,fz,Fz)S(gx_{2n+1},gx_{2n+1},Gx_{2n+1})}{1+S(Fz,Fz,Gx_{2n+1})} \end{array}\right\} + bS(z,z,Gx_{2n+1})$$

letting $n \to \infty$, we have

$$S(Fz,Fz,z) \precsim 2b\ \alpha\ max\left\{\begin{array}{l} S(Fz,Fz,z), S(Fz,Fz,Fz), \\ S(z,z,z), \frac{S(Fz,Fz,Fz)S(z,z,z)}{1+S(Fz,Fz,z)} \end{array}\right\} + bS(z,z,z)$$

$$= 2b\ \alpha\ S(Fz,Fz,z)$$

$$|S(Fz,Fz,z)| \leq 2b\ \alpha\,|S(Fz,Fz,z)|$$

$$(1-2b\ \alpha)\,|S(Fz,Fz,z)| \leq 0.$$

Since $0 < \alpha < \frac{1}{2b}$, we get $|S(Fz,Fz,z)| \leq 0$.

It implies that $|S(Fz,Fz,z)| = 0$. Hence $Fz = z$.

Thus $z = fz = Fz$. (6)

Since the pair (G, g) is weakly compatible, we have $gz = gGv = Ggv = Gz$.

Now

$$S(z, z, Gz)$$
$$= S(Gz, Gz, z)$$
$$\precsim 2bS(Gz, Gz, Fx_{2n+1}) + bS(z, z, Fx_{2n+1})$$
$$= 2bS(Fx_{2n+1}, Fx_{2n+1}, Gz) + bS(z, z, Fx_{2n+1})$$
$$\precsim 2b\ \alpha\ max \left\{ \begin{array}{l} S(fx_{2n+1}, fx_{2n+1}, gz), S(fx_{2n+1}, fx_{2n+1}, Fx_{2n+1}), \\ S(gz, gz, Gz), \frac{S(fx_{2n+1}, fx_{2n+1}, Fx_{2n+1})S(gz,gz,Gz)}{1+S(Fx_{2n+1}, Fx_{2n+1}, Gz)} \end{array} \right\}$$
$$+bS(z, z, Fx_{2n+1})$$

letting $n \to \infty$, we have

$$S(z, z, Gz) = 2b\ \alpha\ \max \left\{ \begin{array}{l} S(z, z, z), S(z, z, z), \\ S(z, z, Gz), \frac{S(z,z,z)S(z,z,Gz)}{1+S(z,z,Gz)} \end{array} \right\} + bS(z, z, z)$$
$$= 2b\ \alpha\ S(z, z, Gz)$$
$$|S(z, z, Gz)| \leq \alpha\, |S(z, z, Gz)|$$
$$(1 - 2b\alpha)\, |S(z, z, Gz)| \leq 0.$$

Since $0 < \alpha < \frac{1}{2b}$, we get $|S(z, z, Gz)| \leq 0$. It imples that $|S(z, z, Gz)| = 0$.

Hence $Gz = gz = z$. (7)

Thus from (6) and (7), z is a common fixed point of F, G, f and g.

For uniqueness,

let $z^* \in X$ be such that $fz^* = Fz^* = z^* = gz^* = Gz^*$.

$$S(z, z, z^*) = S(Fz, Fz, Gz^*)$$
$$\precsim \alpha\ \max \left\{ \begin{array}{l} S(fz, fz, gz^*), S(fz, fz, Fz), \\ S(gz^*, gz^*, Gz^*), \frac{S(fz,fz,Fz)S(gz^*,gz^*,Gz^*)}{1+S(Fz,Fz,Gz^*)} \end{array} \right\}$$

$$= \alpha \ \max \left\{ \begin{array}{l} S(z,z,z^*), S(z,z,z), \\ S(z^*,z^*,z^*), \frac{S(z,z,z)S(z^*,z^*,z^*)}{1+S(z,z,z^*)} \end{array} \right\}$$

$$= \alpha \ S(z,z,z^*)$$

$$|S(z,z,z^*)| \leq \alpha \, |S(z,z,z^*)|$$

$$(1-\alpha) \, |S(z,z,z^*)| \leq 0.$$

Since $0 < \alpha < \frac{1}{2b} < 1$, we get $|S(z,z,z^*)| \leq 0$.

It implies that $|S(z,z,z^*)| = 0$.

Hence $z = z^*$.

Thus z is the unique common fixed point of F, G, f and g.

Now we give an example to illustrate our main Theorem 5.1.2

Example 5.1.3. Let $X = [0,1]$ and $S_b : X^3 \to C$ be defined by $S_b(x,y,z) = |x-z| + i\,|y-z|$. Then X is a complex valued S_b-metric space. Define F, G, f and $g : X \to X$ by $Fx = \frac{x^4}{8^4}$,$Gx = \frac{x^8}{4^8}$,$fx = \frac{x^4}{2^4}$ and $gx = \frac{x^8}{4^4}$ for all $x \in X$. With $\alpha = \frac{1}{11} < 1$.

Consider

$$\begin{aligned} |S(Fx,Fx,Gy)| &= \left| S\left(\tfrac{x^4}{8^4}, \tfrac{x^4}{8^4}, \tfrac{y^8}{4^8} \right) \right| \\ &= \left| \tfrac{x^4}{8^4} - \tfrac{y^8}{4^8} \right| + i \left| \tfrac{x^4}{8^4} - \tfrac{y^8}{4^8} \right| \\ &= \tfrac{1}{16} \left[\left| \tfrac{x^4}{2^4} - \tfrac{y^8}{4^4} \right| + i \left| \tfrac{x^4}{2^4} - \tfrac{y^8}{4^4} \right| \right] \\ &= \tfrac{1}{16} \, |S(fx,fx,gy)| \\ &< \tfrac{1}{11} \, |S(fx,fx,gy)| \\ &= \alpha \, |S(fx,fx,gy)| \\ &\leq \alpha \ \max \left\{ \begin{array}{l} |S(fx,fx,gy)|, |S(fx,fx,Fx)|, |S(gy,gy,Gy)|, \\ \frac{|S(fx,fx,Fx)||S(gy,gy,Gy)|}{|1+S(Fx,Fx,Gy)|} \end{array} \right\}. \end{aligned}$$

Thus (5.1.2.4) is satisfied.

One can easily verify remaining conditions of Theorem 5.1.2.

Clearly $x = 0$ is the unique common fixed point of F, G, f and g.

From Theorem 5.1.2, we have the following corollary.

Corollary 5.1.4. Let (X, S) be a complete complex valued S_b - metric space with coefficient $b > 1$ and $f : X \to X$ be mapping satisfying for all $x, y \in X$

$S(fx, fx, fy) \precsim \alpha\ \max\{S(x, x, y), S(x, x, fx), S(y, y, fy), \frac{S(x,x,fx)S(y,y,fy)}{1+S(fx,fx,fy)}\}$

where $0 < \alpha < 1$. Then f has a unique fixed point.

SECTION 5.2: EXISTENCE AND UNIQUENESS OF COUPLED SUZUKI TYPE RESULT IN S_b METRIC SPACES

In the year 2008, Suzuki[103] generalized the Banach contraction principle [84] as follows.

Theorem 5.2.1. (Suzuki [103]): Let (X, d) be a complete metric space and let T be a mapping on X. Define a non-increasing function $\theta : [0, 1) \to \left(\frac{1}{2}, 1\right]$ by

$$\theta(r) = \begin{cases} 1 & if \;\; 0 \leq r \leq \frac{(\sqrt{5}-1)}{2}, \\ (1-r)r^{-2} & if \;\; \frac{(\sqrt{5}-1)}{2} \leq r \leq \;\; 2^{-\frac{1}{2}}, \\ (1+r)^{-1} & if \;\; 2^{-\frac{1}{2}} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

In the year 2016 S.Sedghi et al.[89] proved the following theorem in S_b-metric spaces.

Theorem 5.2.2. (S.Sedghi et al [89]): Suppose that f, g, M and T are self mappings on a complete S_b-metric space (X, S) such that $f(X) \subseteq T(X)$, $g(X) \subseteq M(X)$. If

$$S(fx, fx, gy) \leq \frac{q}{b^4} \max \left\{ \begin{array}{l} S(Mx, Mx, Ty), S(fx, fx, Mx), S(gy, gy, Ty) \\ \frac{1}{2}[S(Mx, Mx, gy) + S(fx, fx, Ty) \end{array} \right\}$$

holds for each $x, y \in X$ with $0 < q < 1$ and $b \geq \frac{3}{2}$ then f, g, M and T have a unique common fixed point in X provided that M and T are continuous and pairs $\{f, M\}$ and $\{g, T\}$ are compatible.

In this section, we generalize the Theorem 5.2.2 and obtain a Suzuki type common coupled fixed point theorem in S_b-metric spaces. We also furnish an example which supports our main result.

Now we give our main theorem.

Theorem 5.2.3 Let (X, S_b) be a S_b-metric space. Suppose that $C, D : X \times X \to X$ and $P, Q : X \to X$ be satisfying

(5.2.3.1) $C(X \times X) \subseteq Q(X), D(X \times X) \subseteq P(X)$,

(5.2.3.2) $\{C, P\}$ and $\{D, Q\}$ are w-compatible pairs,

(5.2.3.3) one of $P(X)$ or $Q(X)$ is S_b-complete subspace of X,

(5.2.3.4)
$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(C(x,y),C(x,y),Px), S_b(D(u,v),D(u,v),Qu), \\ S_b(C(y,x),C(y,x),Py), S_b(D(v,u),D(v,u),Qv) \end{array}\right\} \leq \max\left\{\begin{array}{c} S_b(Px,Px,Qu), \\ S_b(Py,Py,Qv) \end{array}\right\}$$

implies that

$$\psi\left(S_b(C(x,y),C(x,y),D(u,v))\right) \leq \frac{1}{5b^7}\psi\left(M\left(x,y,u,v\right)\right) - \phi\left(M\left(x,y,u,v\right)\right)$$

for all x, y, u, v in X, where $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ are such that ψ is linear and monotone increasing function and ϕ is lower semi continuous, $\psi(0) = \phi(0) = 0$ and $\phi(t) > 0$, for all $t > 0$ and

$$M\left(x,y,u,v\right) = \max\left\{\begin{array}{c} S_b(Px,Px,Qu), S_b(Py,Py,Qv), \\ S_b(C(x,y),C(x,y),Px), S_b(C(y,x),C(y,x),Py), \\ S_b(D(u,v),D(u,v),Qu), S_b(D(v,u),D(v,u),Qv), \\ \frac{1}{4b^2}\left[S_b(C(x,y),C(x,y),Qu) + S_b(D(u,v),D(u,v),Px)\right], \\ \frac{1}{4b^2}\left[S_b(C(y,x),C(y,x),Qv) + S_b(D(v,u),D(v,u),Py)\right] \end{array}\right\}.$$

Then C, D, P and Q have a unique common coupled fixed point in $X \times X$.

Proof: Let $x_0, y_0 \in X$. From (5.2.3.1), we can construct the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ such that

$$\begin{aligned} C(x_{2n}, y_{2n}) &= Qx_{2n+1} = z_{2n}, \\ C(y_{2n}, x_{2n}) &= Qy_{2n+1} = w_{2n}, \\ D(x_{2n+1}, y_{2n+1}) &= Px_{2n+2} = z_{2n+1}, \\ D(y_{2n+1}, x_{2n+1}) &= Py_{2n+2} = w_{2n+1}, \quad n = 0, 1, 2, \cdots \end{aligned}$$

Case (i): Suppose $z_{2m} = z_{2m+1}$ and $w_{2m} = w_{2m+1}$ for some m.

Assume that $z_{2m+1} \neq z_{2m+2}$ or $w_{2m+1} \neq w_{2m+2}$.

Since

$$\frac{1}{8b^3} \min \left\{ \begin{array}{c} S_b(C(x_{2m+2}, y_{2m+2}), C(x_{2m+2}, y_{2m+2}), Px_{2m+2}), \\ S_b(D(x_{2m+1}, y_{2m+1}), D(x_{2m+1}, y_{2m+1}), Qx_{2m+1}), \\ S_b(C(y_{2m+2}, x_{2m+2}), C(y_{2m+2}, x_{2m+2}), Py_{2m+2}), \\ S_b(D(y_{2m+1}, x_{2m+1}), D(y_{2m+1}, x_{2m+1}), Qy_{2m+1}) \end{array} \right\}$$

$$\leq \max \left\{ S_b(Px_{2m+2}, Px_{2m+2}, Qx_{2m+1}), S_b(Py_{2m+2}, Py_{2m+2}, Qy_{2m+1}) \right\}.$$

From (5.2.3.4), we have

$$\psi\left(S_b(C(x_{2m+2}, y_{2m+2}), C(x_{2m+2}, y_{2m+2}), D(x_{2m+1}, y_{2m+1}))\right)$$

$$\leq \frac{1}{5b^7}\psi\left(M\left(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}\right)\right) - \phi\left(M\left(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}\right)\right)$$

where

$$\begin{aligned} M & \left(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}\right) \\ & = \max \left\{ \begin{array}{l} S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m}), \\ S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}), \\ S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m}), \\ \frac{1}{4b^2}[S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}) + S_b(z_{2m+1}, z_{2m+1}, z_{2m})], \\ \frac{1}{4b^2}[S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) + S_b(w_{2m+1}, w_{2m+1}, w_{2m})] \end{array} \right\} \\ & = \max \left\{ S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \right\}. \end{aligned}$$

Thus

$$\begin{aligned} \psi\left(S_b\left(z_{2m+2}, z_{2m+2}, z_{2m+1}\right)\right) \leq \frac{1}{5b^7}\psi & \left(\max \left\{ \begin{array}{l} S_b\left(z_{2m+2}, z_{2m+2}, z_{2m+1}\right), \\ S_b\left(w_{2m+2}, w_{2m+2}, w_{2m+1}\right) \end{array} \right\} \right) \\ & -\phi \left(\max \left\{ \begin{array}{l} S_b\left(z_{2m+2}, z_{2m+2}, z_{2m+1}\right), \\ S_b\left(w_{2m+2}, w_{2m+2}, w_{2m+1}\right) \end{array} \right\} \right). \end{aligned}$$

Similarly, we can prove

$$\begin{aligned} & \psi\left(S_b\left(w_{2m+2}, w_{2m+2}, w_{2m+1}\right)\right) \\ & \leq \frac{1}{5b^7}\psi\left(\max\left\{ S_b\left(z_{2m+2}, z_{2m+2}, z_{2m+1}\right), S_b\left(w_{2m+2}, w_{2m+2}, w_{2m+1}\right) \right\}\right) \\ & \quad -\phi\left(\max\left\{ S_b\left(z_{2m+2}, z_{2m+2}, z_{2m+1}\right), S_b\left(w_{2m+2}, w_{2m+2}, w_{2m+1}\right) \right\}\right). \end{aligned}$$

It follows that

$$\begin{aligned} & \psi\left(\max\left\{ S_b\left(z_{2m+2}, z_{2m+2}, z_{2m+1}\right), S_b\left(w_{2m+2}, w_{2m+2}, w_{2m+1}\right) \right\}\right) \\ & \leq \frac{1}{5b^7}\psi\left(\max\left\{ S_b\left(z_{2m+2}, z_{2m+2}, z_{2m+1}\right), S_b\left(w_{2m+2}, w_{2m+2}, w_{2m+1}\right) \right\}\right) \\ & \quad -\phi\left(\max\left\{ S_b\left(z_{2m+2}, z_{2m+2}, z_{2m+1}\right), S_b\left(w_{2m+2}, w_{2m+2}, w_{2m+1}\right) \right\}\right). \end{aligned}$$

It follows that $z_{2m+2} = z_{2m+1}$ and $w_{2m+2} = w_{2m+1}$.

Continuing in this process we can conclude that $z_{2m+k} = z_{2m}$ and $w_{2m+k} = w_{2m}$

for all $k \geq 0$.

It follows that $\{z_{2m}\}$ and $\{w_{2m}\}$ are Cauchy sequences.

Case (ii): Assume that $z_{2n} \neq z_{2n+1}$ and $w_{2n} \neq w_{2n+1}$ for all n.

Put $S_n = \max\{S_b(z_{n+1}, z_{n+1}, z_n), S_b(w_{n+1}, w_{n+1}, w_n)\}$.

Since

$$\frac{1}{8b^3} \quad \min \left\{ \begin{array}{c} S_b(C(x_{2n+2}, y_{2n+2}), C(x_{2n+2}, y_{2n+2}), Px_{2n+2}), \\ S_b(D(x_{2n+1}, y_{2n+1}), D(x_{2n+1}, y_{2n+1}), Qx_{2n+1}), \\ S_b(C(y_{2n+2}, x_{2n+2}), C(y_{2n+2}, x_{2n+2}), Py_{2n+2}), \\ S_b(D(y_{2n+1}, x_{2n+1}), D(y_{2n+1}, x_{2n+1}), Qy_{2n+1}) \end{array} \right\}$$

$$\leq \max \left\{ S_b(Px_{2n+2}, Px_{2n+2}, Qx_{2n+1}), S_b(Py_{2n+2}, Py_{2n+2}, Qy_{2n+1}) \right\}.$$

From (5.2.3.4), we have

$$\begin{aligned} \psi\left(S_b(z_{2n+2}, z_{2n+2}, z_{2n+1})\right) \quad &\leq \tfrac{1}{5b^7} \psi\left(M\left(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}\right)\right) \\ &\quad -\phi\left(M\left(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}\right)\right). \end{aligned}$$

Here

$$\begin{aligned} M \quad &(x_{2m+2}, y_{2m+2}, x_{2m+1}, y_{2m+1}) \\ &= \max \left\{ \begin{array}{l} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\ S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}), \\ S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\ \frac{1}{4b^2}[S_b(z_{2n+2}, z_{2n+2}, z_{2n}) + S_b(z_{2n+1}, z_{2n+1}, z_{2n+1})], \\ \frac{1}{4b^2}[S_b(w_{2n+2}, w_{2n+2}, w_{2n}) + S_b(w_{2n+1}, w_{2n+1}, w_{2n+1})] \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), \\ S_b(w_{2n+1}, w_{2n+1}, w_{2n}), S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \end{array} \right\} \\ &= \max \left\{ S_{2n+1}, S_{2n} \right\}. \end{aligned}$$

Therfore

$$\psi\left(S_b(z_{2n+2}, z_{2n+2}, z_{2n+1})\right)$$
$$\leq \tfrac{1}{5b^7}\ \psi\left(\max\left\{\ S_{2n+1}, S_{2n}\ \right\}\right) - \phi\left(\max\left\{\ S_{2n+1}, S_{2n}\ \right\}\right).$$

Similarly, we can prove that

$$\psi\left(S_b(w_{2n+2}, w_{2n+2}, w_{2n+1})\right)$$
$$\leq \tfrac{1}{5b^7}\ \psi\left(\max\left\{\ S_{2n+1}, S_{2n}\ \right\}\right) - \phi\left(\max\left\{\ S_{2n+1}, S_{2n}\ \right\}\right).$$

Thus

$$\psi\left(S_{2n+1}\right)\ \leq \tfrac{1}{5b^7}\ \psi\left(\max\left\{\ S_{2n+1}, S_{2n}\ \right\}\right) - \phi\left(\max\left\{\ S_{2n+1}, S_{2n}\ \right\}\right).$$

If S_{2n+1} is maximum then we get contradiction so that S_{2n} is maximum.

Thus

$$\begin{aligned} \psi\left(S_{2n+1}\right) &\leq \frac{1}{5b^7}\ \psi\left(S_{2n}\right) - \phi\left(S_{2n}\right) \\ &< \psi\left(S_{2n}\right). \end{aligned} \tag{1}$$

Similarly we can conclude that $\psi\left(S_{2n}\right) < \psi\left(S_{2n-1}\right)$.

Since ψ is non - decreasing and continuous, it is clear that $\{S_n\}$ is a non-increasing sequence of non-negative real numbers and must converges to a real number say $k \geq 0$.

Suppose $k > 0$.

Letting $n \to \infty$, in (1), we have $\psi(k) \leq \frac{1}{5b^7}\ \psi(k) - \phi(k) < \psi(k)$.

It is contradiction. Hence $k = 0$

Thus

$$\lim_{n\to\infty} S_b(z_{n+1}, z_{n+1}, z_n) = 0 \tag{2}$$

and

$$\lim_{n\to\infty} S_b(w_{n+1}, w_{n+1}, w_n) = 0. \tag{3}$$

Now we prove that $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences in (X, S).

On contrary we suppose that $\{z_{2n}\}$ or $\{w_{2n}\}$ is not Cauchy. Then there exist $\epsilon > 0$ and monotonically increasing sequence of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that $n_k > m_k$.

$$\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \geq \epsilon \tag{4}$$

and

$$\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_{k-2}}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_{k-2}})\} < \epsilon. \tag{5}$$

From (4) and (5), we have

$$\begin{aligned}
\epsilon \quad \leq \quad & M_k = \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \\
\leq \quad & 2b \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2m_k+2}), S_b(w_{2m_k}, w_{2m_k}, w_{2m_k+2})\} \\
& + b^2 \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k})\} \\
\leq \quad & 4b^2 \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2m_k+1}), S_b(w_{2m_k}, w_{2m_k}, w_{2m_k+1})\} \\
& + 2b^3 \max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k+2}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k+2})\} \\
& + 2b^3 \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1})\} \\
& + b^5 \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_k+1}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_k+1})\}.
\end{aligned}$$

Letting $k \to \infty$ and apply ψ on both sides, we have that

$$\psi\left(\frac{\epsilon}{2b^3}\right) \leq \lim_{k\to\infty} \psi\left(\max\left\{\begin{array}{l} s_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}) \\ s_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1}) \end{array}\right\}\right). \tag{6}$$

Now first we claim that

$$\frac{1}{8b^3} \quad \min\left\{\begin{array}{c} S_b(C(x_{2m_k+2},y_{2m_k+2}),C(x_{2m_k+2},y_{2m_k+2}),Px_{2m_k+2}),\\ S_b(D(x_{2n_k+1},y_{2n_k+1}),D(x_{2n_k+1},y_{2n_k+1}),Qx_{2n_k+1}),\\ S_b(C(y_{2m_k+2},x_{2m_k+2}),C(y_{2m_k+2},x_{2m_k+2}),Py_{2m_k+2}),\\ S_b(D(y_{2n_k+1},x_{2n_k+1}),D(y_{2n_k+1},x_{2n_k+1}),Qy_{2n_k+1}) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} S_b(Px_{2m_k+2},Px_{2m_k+2},Qx_{2n_k+1}),\\ S_b(Py_{2m_k+2},Py_{2m_k+2},Qy_{2n_k+1}) \end{array}\right\}. \tag{7}$$

On contrary suppose that

$$\frac{1}{8b^3} \quad \min\left\{\begin{array}{c} S_b(C(x_{2m_k+2},y_{2m_k+2}),C(x_{2m_k+2},y_{2m_k+2}),Px_{2m_k+2}),\\ S_b(D(x_{2n_k+1},y_{2n_k+1}),D(x_{2n_k+1},y_{2n_k+1}),Qx_{2n_k+1}),\\ S_b(C(y_{2m_k+2},x_{2m_k+2}),C(y_{2m_k+2},x_{2m_k+2}),Py_{2m_k+2}),\\ S_b(D(y_{2n_k+1},x_{2n_k+1}),D(y_{2n_k+1},x_{2n_k+1}),Qy_{2n_k+1}) \end{array}\right\}$$

$$> \max\left\{\ S_b(Px_{2m_k+2},Px_{2m_k+2},Qx_{2n_k+1}),S_b(Py_{2m_k+2},Py_{2m_k+2},Qy_{2n_k+1})\ \right\}.$$

Now from (4), we have

$$\begin{aligned}
\epsilon &\leq \max\{S_b(z_{2m_k},z_{2m_k},z_{2n_k}),S_b(w_{2m_k},w_{2m_k},w_{2n_k})\}\\
&\leq 2b\max\{S_b(z_{2m_k},z_{2m_k},z_{2m_k+1}),S_b(w_{2m_k},w_{2m_k},w_{2m_k+1})\}\\
&\quad +b^2\max\{S_b(z_{2m_k+1},z_{2m_k+1},z_{2n_k}),S_b(w_{2m_k+1},w_{2m_k+1},w_{2n_k})\}\\
&< 2b\max\{S_b(z_{2m_k},z_{2m_k},z_{2m_k+1}),S_b(w_{2m_k},w_{2m_k},w_{2m_k+1})\}\\
&\quad +b^2\frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(z_{2m_k+2},z_{2m_k+2},z_{2m_k+1}),S_b(w_{2m_k+2},w_{2m_k+2},w_{2m_k+1}),\\ S_b(z_{2n_k+1},z_{2n_k+1},z_{2n_k}),S_b(w_{2n_k+1},w_{2n_k+1},w_{2n_k}) \end{array}\right\}.
\end{aligned}$$

Letting $k\to\infty$, we have $\epsilon \leq 0$. It is a contradiction.

Hence the claim is holds that is (7) holds.

Now from (5.2.3.4), we have

$$\psi\left(S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1})\right)$$

$$\leq \frac{1}{5b^7}\ \psi\left(\max\left\{\begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{1}{4b^2}[S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})], \\ \frac{1}{4b^2}[S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})] \end{array}\right\}\right)$$

$$-\phi\left(\max\left\{\begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{1}{4b^2}[S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})], \\ \frac{1}{4b^2}[S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})] \end{array}\right\}\right).$$

Similarly

$$\psi\quad\left(S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1})\right)$$

$$\leq \frac{1}{5b^7}\ \psi\left(\max\left\{\begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{1}{4b^2}[S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})], \\ \frac{1}{4b^2}[S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})] \end{array}\right\}\right)$$

$$-\phi\left(\max\left\{\begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{1}{4b^2}[S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})], \\ \frac{1}{4b^2}[S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})] \end{array}\right\}\right).$$

Thus

$$\psi\left(\max\left\{ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}) \right\}\right)$$

$$\leq \frac{1}{5b^7}\,\psi\left(\max\left\{\begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{1}{4b^2}\left[\begin{array}{l} S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) \\ +S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1}) \end{array}\right], \\ \frac{1}{4b^2}\left[\begin{array}{l} S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) \\ +S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1}) \end{array}\right] \end{array}\right\}\right)$$

$$-\phi\left(\max\left\{\begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{1}{4b^2}\left[\begin{array}{l} S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) \\ +S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1}) \end{array}\right], \\ \frac{1}{4b^2}\left[\begin{array}{l} S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) \\ +S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1}) \end{array}\right] \end{array}\right\}\right) \tag{8}$$

But

$$\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\}$$

$$\begin{aligned}
&\leq 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad +b^2\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \\
&\leq 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad +b^2\left(2b\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_{k-2}}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_{k-2}})\}\right) \\
&\quad +b^2\left(b^2\max\{S_b(z_{2n_k-2}, z_{2n_k-2}, z_{2n_k}), S_b(w_{2n_k-2}, w_{2n_k-2}, w_{2n_k})\}\right) \\
&\leq 2b^2\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2m_k+1}), S_b(w_{2m_k}, w_{2m_k}, w_{2m_k+1})\} \\
&\quad +2b^3\epsilon + 2b^5\max\{S_b(z_{2n_k-1}, z_{2n_k-1}, z_{2n_k}), S_b(w_{2n_k-1}, w_{2n_k-1}, w_{2n_k})\} \\
&\quad +b^6\max\{S_b(z_{2n_k-1}, z_{2n_k-1}, z_{2n_k}), S_b(w_{2n_k-1}, w_{2n_k-1}, w_{2n_k})\}.
\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty}\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\} \leq 2b^3\epsilon.$$

Also

$$\begin{aligned}
&\lim_{k\to\infty}\tfrac{1}{4b^2}[S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})] \\
&\leq \lim_{k\to\infty}\tfrac{1}{4b^2}\begin{bmatrix} 2bS_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}) + b^2S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}) + \\ 2bS_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}) + b^2S_b(z_{2n_k}, z_{2n_k}, z_{2m_k+1}) \end{bmatrix} \\
&\leq \lim_{k\to\infty}\tfrac{1}{4b^2}[b^3S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}) + b^2S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})] \\
&\leq \tfrac{1}{4b^2}[2b^6\epsilon + 2b^5\epsilon] \\
&\leq \tfrac{(1+b)2b^5\epsilon}{4b^2} \\
&= b^4\epsilon.
\end{aligned}$$

Similarly

$$\lim_{k\to\infty}\frac{1}{4b^2}\left[S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})\right] \leq b^4\epsilon.$$

Letting $k \to \infty$ in (8), we have

$$\lim_{k\to\infty} \psi \left(\max \left\{ \begin{array}{c} S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}) \end{array} \right\} \right)$$
$$\leq \tfrac{1}{5b^7}\psi\left(\max\{2b^3\epsilon, 0, 0, 0, 0, b^4\epsilon, b^4\epsilon\}\right)$$
$$- \lim_{k\to\infty} \phi \left(\max \left\{ \begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{1}{4b^2} S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1}), \\ \frac{1}{2b} S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1}) \end{array} \right\} \right)$$
$$\leq \tfrac{1}{5b^7}\psi\left(\max\{2b^3\epsilon, b^4\epsilon\}\right).$$

Therefore

$$\lim_{k\to\infty} \psi \left(\max \left\{ \begin{array}{l} S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), \\ S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}) \end{array} \right\} \right) \leq \tfrac{1}{5b^7}\psi\left(\max\{2b^3\epsilon, b^4\epsilon\}\right). (9)$$

Now letting $n \to \infty$ in (6), from (2), (3) and (9), we have

$$\psi\left(\frac{\epsilon}{2b^3}\right) \leq \frac{1}{5b^7}\psi\left(\max\{2b^3\epsilon, b^4\epsilon\}\right).$$

$Subcase(i)$: If $2b^3\epsilon$ is maximum, by the definition of ψ, we have that

$$b^2 \leq \frac{4}{5}.$$

It is a contradiction.

$Subcase(ii)$: If $b^4\epsilon$ is maximum, by the definition of ψ, we have that

$$b \leq \frac{2}{5}.$$

It is a contradiction.

Hence $\{z_{2n}\}$ and $\{w_{2n}\}$ are S_b-Cauchy sequences in (X, S).

In addition

$\max\{S_b(z_{2n+1}, z_{2n+1}, z_{2m+1}), S_b(w_{2n+1}, w_{2n+1}, w_{2m+1})\}$

$$\begin{aligned}
&\leq 2b \max\{S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n})\} \\
&\quad + b^2 \max\{S_b(z_{2n}, z_{2n}, z_{2m+1}), S_b(w_{2n}, w_{2n}, w_{2m+1})\} \\
&\leq 2b^2 \max\{S_b(z_{2n}, z_{2n}, z_{2n+1}), S_b(w_{2n}, w_{2n}, w_{2n+1})\} \\
&\quad + 2b^3 \max\{S_b(z_{2n}, z_{2n}, z_{2m}), S_b(w_{2n}, w_{2n}, w_{2m})\} \\
&\quad + b^4 \max\{S_b(z_{2m}, z_{2m}, z_{2m+1}), S_b(w_{2m}, w_{2m}, w_{2m+1})\}.
\end{aligned}$$

It is clear that

$$S_b(z_{2n+1}, z_{2n+1}, z_{2m+1}) < \epsilon \text{ as n,m} \to \infty$$

and

$$S_b(w_{2n+1}, w_{2n+1}, w_{2m+1}) < \epsilon \text{ as n,m} \to \infty.$$

Therefore $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are also S_b-Cauchy sequences in (X, S).

Hence $\{z_n\}$ and $\{w_n\}$ are S_b-Cauchy sequences in (X, S).

Suppose $P(X)$ is S_b- complete subspace of (X, S). Then the sequences $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are converges to α and β in $P(X)$. Thus there exist a and b in $P(X)$ such that

$$\lim_{n\to\infty} z_n = \alpha = Pa \text{ and } \lim_{n\to\infty} w_n = \beta = Pb. \tag{10}$$

Before going to proving common coupled fixed point for the mappings C, D, P and Q, first we claim that for each $n \geq 1$ at least one of the following assertion is holds.

$$\frac{1}{8b^3} \min\left\{\begin{array}{c} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), \\ S_b(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array}\right\} \leq \max\left\{ S_b(\alpha, \alpha, z_{2n}), S_b(\beta, \beta, w_{2n}) \right\}$$

or

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(z_{2n},z_{2n},z_{2n-1}), \\ S_b(w_{2n},w_{2n},w_{2n-1}) \end{array}\right\} \leq \max\left\{ S_b(\alpha,\alpha,z_{2n-2}), S_b(\beta,\beta,w_{2n-2}) \right\}.$$

On contrary suppose that

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(z_{2n+1},z_{2n+1},z_{2n}), \\ S_b(w_{2n+1},w_{2n+1},w_{2n}) \end{array}\right\} > \max\left\{ S_b(\alpha,\alpha,z_{2n}), S_b(\beta,\beta,w_{2n}) \right\}$$

and

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(z_{2n},z_{2n},z_{2n-1}), \\ S_b(w_{2n},w_{2n},w_{2n-1}) \end{array}\right\} > \max\left\{ S_b(\alpha,\alpha,z_{2n-1}), S_b(\beta,\beta,w_{2n-1}) \right\}.$$

Now consider

$$\begin{aligned}
&\min\left\{\begin{array}{c} S_b(z_{2n},z_{2n},z_{2n-1}), \\ S_b(w_{2n},w_{2n},w_{2n-1}) \end{array}\right\} \\
&\leq \min\left\{\begin{array}{c} 2bS_b(z_{2n},z_{2n},\alpha)+b^2S_b(\alpha,\alpha,z_{2n-1}), \\ 2bS_b(w_{2n},w_{2n},\beta)+b^2S_b(\beta,\beta,z_{2n-1}) \end{array}\right\} \\
&\leq 2b^2\max\left\{\begin{array}{c} S_b(\alpha,\alpha,z_{2n}), \\ S_b(\beta,\beta,w_{2n}) \end{array}\right\} + b^2\max\left\{\begin{array}{c} S_b(\alpha,\alpha,z_{2n-1}), \\ S_b(\beta,\beta,z_{2n-1}) \end{array}\right\} \\
&< \frac{1}{4b}\min\left\{\begin{array}{c} S_b(z_{2n+1},z_{2n+1},z_{2n}), \\ S_b(w_{2n+1},w_{2n+1},w_{2n}) \end{array}\right\} + \frac{1}{8b}\min\left\{\begin{array}{c} S_b(z_{2n},z_{2n},z_{2n-1}), \\ S_b(w_{2n},w_{2n},w_{2n-1}) \end{array}\right\} \\
&\leq \frac{1}{4b}\min\left\{\begin{array}{c} S_b(z_{2n},z_{2n},z_{2n-1}), \\ S_b(w_{2n},w_{2n},w_{2n-1}) \end{array}\right\} + \frac{1}{8b}\min\left\{\begin{array}{c} S_b(z_{2n},z_{2n},z_{2n-1}), \\ S_b(w_{2n},w_{2n},w_{2n-1}) \end{array}\right\} \\
&= \frac{3}{8b}\min\left\{ S_b(z_{2n},z_{2n},z_{2n-1}), S_b(w_{2n},w_{2n},w_{2n-1}) \right\}.
\end{aligned}$$

It is a contradiction.

Hence our assertion is holds.

Sub case(a): $\frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), \\ S_b(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array}\right\} \leq \max\left\{ S_b(\alpha,\alpha,z_{2n}), S_b(\beta,\beta,w_{2n}) \right\}$

is holds.

Now we have to prove that $C(a,b)=\alpha$ and $C(b,a)=\beta$.

On contrary suppose that $C(a,b)\neq\alpha$ or $C(b,a)\neq\beta$.

Since

$$\begin{aligned} &\frac{1}{8b^3}\min\left\{\begin{array}{l} S_b(C(a,b),C(a,b),\alpha), S_b(z_{2n+1},z_{2n+1},z_{2n}), \\ S_b(C(b,a),C(b,a),\beta), S_b(w_{2n+1},w_{2n+1},w_{2n}) \end{array}\right\} \\ &\leq \max\left\{S_b(\alpha,\alpha,z_{2n}), S_b(\beta,\beta,w_{2n})\right\} \end{aligned}$$

From (5.2.3.4), by deinition of ψ and Lemma 1.9.9(Ch-1), we have

$$\begin{aligned} &\psi\left(\tfrac{1}{4b^2}S_b\left(C(a,b),C(a,b),\alpha\right)\right) \\ &\leq \lim_{n\to\infty}\inf\psi\left(S_b\left(C(a,b),C(a,b),D(x_{2n+1},y_{2n+1})\right)\right) \\ &\leq \frac{1}{5b^7}\lim_{n\to\infty}\inf\psi\left(\max\left\{\begin{array}{l} S_b(\alpha,\alpha,z_{2n}), S_b(\beta,\beta,w_{2n}), \\ S_b\left(C(a,b),C(a,b),\alpha\right), \\ S_b\left(C(b,a),C(b,a),\beta\right), \\ \frac{1}{2b}\left[\begin{array}{l} S_b\left(C(a,b),C(a,b),Qx_{2n+1}\right)+ \\ S_b\left(z_{2n+1},z_{2n+1},\alpha\right) \end{array}\right], \\ \frac{1}{4b^2}\left[\begin{array}{l} S_b\left(C(b,a),C(b,a),z_{2n}\right)+ \\ S_b\left(w_{2n+1},w_{2n+1},\beta\right) \end{array}\right] \end{array}\right\}\right) \end{aligned}$$

$$- \lim_{n\to\infty} \inf \phi \left(\max \left\{ \begin{array}{l} S_b(\alpha, \alpha, z_{2n}), S_b(\beta, \beta, w_{2n}), \\ S_b\left(C(a,b), C(a,b), \alpha\right), \\ S_b\left(C(b,a), C(b,a), \beta\right), \\ \frac{1}{2b} \left[\begin{array}{l} S_b\left(C(a,b), C(a,b), Qx_{2n+1}\right) + \\ S_b\left(z_{2n+1}, z_{2n+1}, \alpha\right) \end{array} \right], \\ \frac{1}{4b^2} \left[\begin{array}{l} S_b\left(C(b,a), C(b,a), z_{2n}\right) + \\ S_b\left(w_{2n+1}, w_{2n+1}, \beta\right) \end{array} \right] \end{array} \right\} \right)$$

$$= \tfrac{1}{5b^7} \psi \left(\max \left\{ S_b\left(C(a,b), C(a,b), \alpha\right), S_b\left(C(b,a), C(b,a), \beta\right) \right\}\right)$$

Similarly

$$\psi\left(\tfrac{1}{4b^2} S_b\left(C(b,a), C(b,a), \beta\right)\right)$$
$$\leq \tfrac{1}{5b^7} \psi \left(\max \left\{ \; S_b\left(C(b,a), C(b,a), \alpha\right) S_b\left(C(b,a), C(b,a), \beta\right) \right\} \right)$$

Thus

$$\psi \left(\tfrac{1}{4b^2} \max \left\{ \begin{array}{c} S_b(C(a,b), C(a,b), \alpha), \\ S_b(C(b,a), C(b,a), \beta) \end{array} \right\} \right) \leq \tfrac{1}{5b^7} \; \psi \left(\max \left\{ \begin{array}{c} S_b(C(a,b), C(a,b), \alpha), \\ S_b(C(b,a), C(b,a), \beta) \end{array} \right\} \right).$$

By the definition of ψ, it follows that $C(a,b) = \alpha = Pa$ and $C(b,a) = \beta = Pb$.

Since (C, P) is w-compatible pair, we have $C(\alpha, \beta) = P\alpha$ and $C(\beta, \alpha) = P\beta$.

From the definition of S_b-metric it is clear that

$$\tfrac{1}{8b^3} \min \left\{ \begin{array}{c} S_b(C(\alpha,\beta), C(\alpha,\beta), P\alpha), S_b(C(\beta,\alpha), C(\beta,\alpha), P\beta) \\ S_b(D(x_{2n+1}, y_{2n+1}), D(x_{2n+1}, y_{2n+1}), Qx_{2n+1}), \\ S_b(D(y_{2n+1}, x_{2n+1}), D(y_{2n+1}, x_{2n+1}), Qy_{2n+1}) \end{array} \right\}$$

$$\leq \max \left\{ \; S_b(P\alpha, P\alpha, Qx_{2n+1}), S_b(P\beta, P\beta, Qy_{2n+1}) \right\}.$$

From (5.2.3.4), by the definition of ψ and Lemma 1.9.9(Ch-1), we have

$$\begin{aligned}
\psi & \left(\tfrac{1}{4b^2} S_b(C(\alpha,\beta), C(\alpha,\beta), \alpha)\right) \\
& \leq \limsup_{n\to\infty} \psi\left(S_b(C(\alpha,\beta), C(\alpha,\beta), D(x_{2n+1}, y_{2n+1}))\right) \\
& \leq \frac{1}{5b^7} \limsup_{n\to\infty} \psi \left(\max \left\{ \begin{array}{c} S_b(C(\alpha,\beta), C(\alpha,\beta), z_{2n}), S_b(C(\beta,\alpha), C(\beta,\alpha), w_{2n}), \\ S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\ S_b(z_{2n+1}, z_{2n+1}, C(\alpha,\beta)), S_b(w_{2n+1}, w_{2n+1}, C(\beta,\alpha)), \end{array} \right\} \right) \\
& \qquad - \limsup_{n\to\infty} \phi \left(\max \left\{ \begin{array}{c} S_b(C(\alpha,\beta), C(\alpha,\beta), z_{2n}), S_b(C(\beta,\alpha), C(\beta,\alpha), w_{2n}), \\ S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\ S_b(z_{2n+1}, z_{2n+1}, C(\alpha,\beta)), S_b(w_{2n+1}, w_{2n+1}, C(\beta,\alpha)), \end{array} \right\} \right) \\
& \leq \frac{1}{5b^7} \psi \left(\max \left\{ \begin{array}{c} 2bS_b(C(\alpha,\beta), C(\alpha,\beta), \alpha), 2bS_b(C(\beta,\alpha), C(\beta,\alpha), \beta), \\ 0, 0, b^2 S_b(\alpha, \alpha, C(\alpha,\beta)), b^2 S_b(\beta, \beta, C(\beta,\alpha)), \end{array} \right\} \right) \\
& \leq \frac{1}{5b^7} \psi \left(2b^2 \max \left\{ S_b(C(\alpha,\beta), C(\alpha,\beta), \alpha), S_b(C(\beta,\alpha), C(\beta,\alpha), \beta) \right\} \right).
\end{aligned}$$

Similarly

$$\psi\left(\tfrac{1}{4b^2} S_b(C(\beta,\alpha), C(\beta,\alpha), \beta)\right) \leq \frac{1}{5b^7} \psi \left(2b^2 \max \left\{ \begin{array}{c} S_b(C(\alpha,\beta), C(\alpha,\beta), \alpha), \\ S_b(C(\beta,\alpha), C(\beta,\alpha), \beta) \end{array} \right\} \right).$$

Thus

$$\psi \left(\frac{1}{4b^2} \max \left\{ \begin{array}{c} S_b(C(\alpha,\beta), C(\alpha,\beta), \alpha), \\ S_b(C(\beta,\alpha), C(\beta,\alpha), \beta) \end{array} \right\} \right) \leq \frac{1}{5b^7} \psi \left(2b^2 \max \left\{ \begin{array}{c} S_b(C(\alpha,\beta), C(\alpha,\beta), \alpha), \\ S_b(C(\beta,\alpha), C(\beta,\alpha), \beta) \end{array} \right\} \right).$$

By the definition of ψ, it follows that

$C(\alpha,\beta) = \alpha = P\alpha$ and $C(\beta,\alpha) = \beta = P\beta$.

Therefore (α, β) is common coupled fixed point of C and P.

Since $C(X \times X) \subseteq Q(X)$ there exist x and y in X such that

$C(\alpha, \beta) = \alpha = Qx$ and $C(\beta, \alpha) = \beta = Qy$.

Since we have that

$$\frac{1}{8b^3} \min \left\{ \begin{array}{c} S_b(C(\alpha,\beta), C(\alpha,\beta), P\alpha), S_b(C(\beta,\alpha), C(\beta,\alpha), P\beta) \\ S_b(D(x,y), D(x,y), Qx), S_b(D(y,x), D(y,x), Qy) \end{array} \right\} \leq \max \left\{ \begin{array}{c} S_b(P\alpha, P\alpha, Qx), \\ S_b(P\beta, P\beta, Qy) \end{array} \right\}.$$

From (5.2.3.4), we have

$$\psi \quad (S(C(\alpha,\beta), C(\alpha,\beta), D(x,y))) \leq \frac{1}{5b^7}\, \psi \left(b \max \left\{ \begin{array}{c} S_b(\alpha,\alpha,D(x,y)), \\ S_b(\beta,\beta,D(y,x)) \end{array} \right\} \right) - \phi \left(\max \left\{ \begin{array}{c} S_b(D(x,y), D(x,y), \alpha), \\ S_b(D(y,x), D(y,x), \beta) \end{array} \right\} \right).$$

Similarly

$$\psi\left(S_b(\beta,\beta,D(y,x))\right) \leq \frac{1}{5b^7}\, \phi \left(b \max \left\{ \begin{array}{c} S_b(\alpha,\alpha,D(x,y)), \\ S_b(\beta,\beta,D(y,x)) \end{array} \right\} \right) - \phi \left(\max \left\{ \begin{array}{c} S_b(D(x,y), D(x,y), \alpha), \\ S_b(D(y,x), D(y,x), \beta) \end{array} \right\} \right).$$

Thus

$$\psi \left(\max \left\{ \begin{array}{c} S_b(\alpha,\alpha,D(x,y)), \\ S_b(\beta,\beta,D(y,x)) \end{array} \right\} \right) \leq \frac{1}{5b^7}\, \phi \left(b \max \left\{ \begin{array}{c} S_b(\alpha,\alpha,D(x,y)), \\ S_b(\beta,\beta,D(y,x)) \end{array} \right\} \right) - \phi \left(\max \left\{ \begin{array}{c} S_b(D(x,y), D(x,y), \alpha), \\ S_b(D(y,x), D(y,x), \beta) \end{array} \right\} \right).$$

It follows that $D(x, y) = \alpha = Qx$ and $D(y, x) = \beta = Qy$.

Since (D, Q) is w-compatible pair, we have $D(\alpha, \beta) = Q\alpha$ and $D(\beta, \alpha,) = Q\beta$.

Since we have that

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(C(\alpha,\beta),C(\alpha,\beta),P\alpha),S_b(C(\beta,\alpha),C(\beta,\alpha),P\beta) \\ S_b(D(\alpha,\beta),D(\alpha,\beta),Q\alpha),S_b(D(\beta,\alpha),D(\beta,\alpha),Q\beta) \end{array}\right\}$$
$$\leq \max\left\{\begin{array}{c} S_b(P\alpha,P\alpha,Q\alpha), \\ S_b(P\beta,P\beta,Q\beta) \end{array}\right\}.$$

From (5.2.3.4) we have

$$\begin{aligned}\psi\ &(S_b(C(\alpha,\beta),C(\alpha,\beta),D(\alpha,\beta)))\\ &\leq \frac{1}{5b^7}\ \psi\left(\max\left\{\begin{array}{c} S_b(\alpha,\alpha,D(\alpha,\beta)),S_b(\beta,\beta,D(\beta,\alpha)), \\ S_b(D(\alpha,\beta),D(\alpha,\beta),\alpha),S_b(D(\beta,\alpha),D(\beta,\alpha),\beta) \end{array}\right\}\right)\\ &\quad -\phi\left(\max\left\{\begin{array}{c} S_b(\alpha,\alpha,D(\alpha,\beta)),S_b(\beta,\beta,D(\beta,\alpha)), \\ S_b(D(\alpha,\beta),D(\alpha,\beta),\alpha),S_b(D(\beta,\alpha),D(\beta,\alpha),\beta) \end{array}\right\}\right)\\ &\leq \frac{1}{5b^7}\ \psi\left(b\max\left\{ S_b(\alpha,\alpha,D(\alpha,\beta)),S_b(\beta,\beta,D(\beta,\alpha)) \right\}\right).\end{aligned}$$

Similarly

$$\begin{aligned}&\psi\left(S_b(\beta,\beta,D(\beta,\alpha))\right)\\ &\qquad\leq \frac{1}{5b^7}\ \psi\left(b\max\left\{ S_b(\alpha,\alpha,D(\alpha,\beta)),S_b(\beta,\beta,D(\beta,\alpha)) \right\}\right).\end{aligned}$$

Thus

$$\psi\left(\max\left\{\begin{array}{c} S_b(\alpha,\alpha,D(\alpha,\beta)), \\ S_b(\beta,\beta,D(\beta,\alpha)) \end{array}\right\}\right) \leq \frac{1}{5b^7}\ \psi\left(b\max\left\{\begin{array}{c} S_b(\alpha,\alpha,D(\alpha,\beta)), \\ S_b(\beta,\beta,D(\beta,\alpha)) \end{array}\right\}\right).$$

It follows that $D(\alpha,\beta)=\alpha=Q\alpha$ and $D(\beta,\alpha)=\beta=Q\beta$.

Therefore (α,β) is common coupled fixed point of C,D,P and Q.

To prove uniqueness let us take (α^1,β^1) is another common coupled fixed point of C, D, P and Q.

Since it is clear that

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(C(\alpha,\beta),C(\alpha,\beta),P\alpha), S_b(C(\beta,\alpha),C(\beta,\alpha),P\beta), \\ S_b(D(\alpha^1,\beta^1),D(\alpha^1,\beta^1),Q\alpha^1), S_b(D(\beta^1,\alpha^1),D(\beta^1,\alpha^1),Q\beta^1) \end{array}\right\}$$
$$\leq \max\left\{\begin{array}{c} S_b(P\alpha,P\alpha,Q\alpha^1), \\ S_b(P\beta,P\beta,Q\beta^1) \end{array}\right\}.$$

From (5.2.3.4), we have

$$\psi\left(S_b(\alpha,\alpha,\alpha^1)\right)$$
$$=\psi\left(S_b(C(\alpha,\beta),C(\alpha,\beta),D(\alpha^1,\beta^1))\right)$$
$$\leq \frac{1}{5b^7}\ \psi\left(\max\left\{\begin{array}{c} S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1), S_b(\alpha,\alpha,\alpha), \\ S_b(\beta,\beta,\beta), S_b(\alpha^1,\alpha^1,\alpha^1), S_b(\beta^1,\beta^1,\beta^1), \\ \frac{1}{4b^2}\left[S_b(\alpha,\alpha,\alpha^1)+S_b(\alpha^1,\alpha^1,\alpha)\right], \frac{1}{4b^2}\left[S_b(\beta,\beta,\beta^1)+S_b(\beta^1,\beta^1,\beta)\right] \end{array}\right\}\right)$$

$$-\phi\left(\max\left\{\begin{array}{c} S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1), S_b(\alpha,\alpha,\alpha), \\ S_b(\beta,\beta,\beta), S_b(\alpha^1,\alpha^1,\alpha^1), S_b(\beta^1,\beta^1,\beta^1), \\ \frac{1}{4b^2}\left[S_b(\alpha,\alpha,\alpha^1)+S_b(\alpha^1,\alpha^1,\alpha)\right], \frac{1}{4b^2}\left[S_b(\beta,\beta,\beta^1)+S_b(\beta^1,\beta^1,\beta)\right] \end{array}\right\}\right)$$

$$\leq \frac{1}{5b^7}\ \psi(b\max\{S_b(\alpha,\alpha,\alpha^1),S_b(\beta,\beta,\beta^1)\}).$$

Similarly

$$\psi\left(S_b(\beta,\beta,\beta^1)\right)\leq \frac{1}{5b^7}\ \psi(b\max\{S_b(\alpha,\alpha,\alpha^1),S_b(\beta,\beta,\beta^1)\}).$$

Thus

$$\psi\left(\max\left\{\ S_b(\alpha,\alpha,\alpha^1),S_b(\beta,\beta,\beta^1)\ \right\}\right)\leq \frac{1}{5b^7}\ \psi(b\max\{S_b(\alpha,\alpha,\alpha^1),S_b(\beta,\beta,\beta^1)\}).$$

It follows that $\alpha = \alpha^1$ and $\beta = \beta^1$.

Hence (α, β) is unique common coupled fixed point of C, D, P and Q.

Similarly the remaining proof also follows when the Sub case(b) holds.

$$i.e.\ \frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(z_{2n}, z_{2n}, z_{2n-1}), \\ S_b(w_{2n}, w_{2n}, w_{2n-1}) \end{array}\right\} \leq \max\left\{ S_b(\alpha, \alpha, z_{2n-1}), S_b(\beta, \beta, w_{2n-1}) \right\}$$

is holds.

Now we give an example to illustrate our main theorem.

Example 5.2.4. Let $X = [0, 1]$ and $S : X \times X \times X \to \mathbb{R}^+$ by

$S_b(x, y, z) = (|y + z - 2x| + |y - z|)^2$, then S is S_b metric space with $b = 4$.

Define $C, D : X \times X \to X$ and $P, Q : X \to X$ by $C(x, y) = \frac{x+y}{4^7\sqrt{3}}$,

$D = \frac{x+y}{4^8\sqrt{3}}$, $P(x) = \frac{x}{4}$ and $Q(x) = \frac{x}{16}$. Also define $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = t$ and $\phi(t) = \frac{t}{30b^7}$.

$$\begin{aligned}
&\psi\left(S_b(C(x, y), C(x, y), D(u, v))\right) \\
&= (|C(x, y) + D(u, v) - 2C(x, y)| + |C(x, y) - D(u, v)|)^2 \\
&= (2\ |C(x, y) - D(u, v)|)^2 \\
&= 4\left|\frac{x+y}{4^7\sqrt{3}} - \frac{u+v}{4^8\sqrt{3}}\right|^2 \\
&= \frac{2}{3}\ \left|\frac{4x-u}{4^8} + \frac{4y-v}{4^8}\right|^2 \\
&\leq \frac{1}{6(4^5)^2}\ \left(\max\left\{\left|\frac{4x^2-u^2}{16}\right|, \left|\frac{4y^2-v^2}{16}\right|\right\}\right)^2 \\
&\leq \frac{1}{6(4^{10})}\ \max\left\{\left|\frac{x}{4} - \frac{u}{16}\right|^2, \left|\frac{y}{4} - \frac{v}{16}\right|^2\right\} \\
&= \frac{1}{6(4^{10})}\ \max\left\{ S_b(Px, Px, Qu), S_b(Py, Py, Qv), S_b(C(x, y), C(x, y), Px) \right\} \\
&\leq \frac{1}{5b^7}\ \psi\left(M\left(x, y, u, v\right)\right) - \phi\left(M\left(x, y, u, v\right)\right)
\end{aligned}$$

Thus the condition (5.2.3.4) is satisfied.One can easily verify remaining conditions of Theorem 5.2.3 and $(0, 0)$ is unique common coupled fixed point of C, D, P and Q.

Theorem 5.2.5. Let (X, S) be a complete S_b-metric space. Suppose that $A : X \times X \to X$ be mapping satisfying

$$\frac{1}{8b^3}\min\left\{\begin{array}{l} S(A(x,y),A(x,y),x), \\ S(A(u,v),A(u,v),u), \end{array}\right\} \leq \max\left\{ S(x,x,u), S(y,y,v) \right\}$$

implies that

$$\psi\left(S(A(x,y),A(x,y),A(u,v))\right) \leq \frac{1}{5b^7}\psi\left(M\left(x,y,u,v\right)\right) - \phi\left(M\left(x,y,u,v\right)\right)$$

for all x, y, u, v in X, where $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ are such that ψ is linear and monotonically increasing function and ϕ is lower semi continuous, $\psi(0) = \phi(0) = 0$ and $\phi(t) > 0$, for all $t > 0$ and

$$M\left(x,y,u,v\right) = \max\left\{\begin{array}{c} S(x,x,u), S(y,y,v), S(A(x,y),A(x,y),x), \\ S(A(y,x),A(y,x),y), S(A(u,v),A(u,v),u), \\ S(A(v,u),A(v,u),v), \\ \frac{1}{4b^2}\left[S(A(x,y),A(x,y),u) + S(A(u,v),A(u,v),x)\right], \\ \frac{1}{4b^2}\left[S(A(y,x),A(y,x),v) + S(A(v,u),A(v,u),y)\right] \end{array}\right\}.$$

Then A has a unique coupled fixed point in $X \times X$.

CHAPTER 6

A NEW COMMON COUPLED FIXED POINT RESULT FOR CONTRACTIVE MAPS INVOLVING DOMINATING FUNCTIONS

In this chapter we establish a new common coupled fixed point theorem for contractive inequalities using an auxiliary function which dominate the ordinary metric function.

Now we extend the Salimi et al.[77] Definition 1.11.3(Ch-1) to Jungck type maps of which one is a coupled map as follows.

Definition 6.1. Let (X, d) be a metric space and $F : X \times X \to X$ and $g : X \to X$ be mappings. Let $\alpha : X \times X \to \mathbb{R}^+$. The pair (F, g) is said to be α-admissible with respect to d if $x, y \in X, \alpha(gx, gy) \geq d(gx, gy)$ implies $\alpha(F(x, x), F(y, y)) \geq d(F(x, x), F(y, y))$.

Definition 6.2. Let X be a non-empty set, $F : X \times X \to X$ and $g : X \to X$ be mappings.

(*i*) ([107]) An element $(x, y) \in X \times X$ is called a coupled coincidence point of F and g if $gx = F(x, y)$ and $gy = F(y, x)$.

(*ii*) ([107]) An element $(x, y) \in X \times X$ is called a common coupled fixed point of F and g if $gx = x = F(x, y)$ and $gy = y = F(y, x)$.

(*iii*) ([56]) The pair (F, g) is w-compatible if $g(F(x, y)) = F(gx, gy)$ and $g(F(y, x)) = F(gy, gx)$ whenever there exist $x, y \in X$ with $gx = F(x, y)$ and $gy = F(y, x)$.

(iv) The pair (F, g) is commuting if $g(F(x, y)) = F(gx, gy)$ and $g(F(y, x)) = F(gy, gx)$.

Definition 6.3. Let Φ be the family of non-decreasing and continuous functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t > 0$. Clearly $\phi(t) < t$ for $t > 0$ and $\phi(0) = 0$.

In 2016 N.Hussain et al.[69] proved the following theorem.

Theorem 6.4.(N.Hussain et al.[69]): Let $\alpha : X \times X \to \mathbb{R}^+$ be a mapping and (X, d) be a complete metric sapce. Let T be a self-mapping on X and the following assertions hold.

(i) T is α-admissible mapping with respaect to d,

(ii) either T is continuous or,

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, Tx) \geq d(x, Tx)$ for all $n \in \mathbb{N} \bigcup \{0\}$ and $\lim_{n, \to\infty} \alpha(x_n, Tx) \geq d(x, Tx)$,

(iv) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq d(x_0, Tx_0)$,

(v) there exists $\psi \in \Psi$ such that for all $x, y \in X$,

$$\alpha(Tx, Ty) \leq \psi(\alpha(x, y)).$$

Then T has a fixed point.

We observed that the authors inherently used the continuity of ψ when using (iii) and (v).

In this chapter we generalize the N.Hussain et al.[69] Theorem 6.4 and obtain a new common couled fixed point theroem.

Now we give our main Theorem.

Theorem 6.5. Let (X,d) be a metric space, $F : X \times X \to X$ and $g : X \to X$ be mapping. Let $\alpha : X \times X \to \mathbb{R}^+$ be a mapping with $\alpha(x,y) = 0 \Leftrightarrow x = y$. Assume

(6.5.1) $F(X \times X) \subseteq g(X)$, $g(X)$ is a complete sub space of X,

(6.5.2) the pair (F,g) is commuting,

(6.5.3) the pair (F,g) is α-admissible with respect to d,

(6.5.4) $\alpha(gx_0, F(x_0,x_0)) \geq d(gx_0, F(x_0,x_0))$ for some $x \in X$.

(6.5.5)

$$\alpha(F(x,y), F(u,v) \leq \phi\left(\max\left\{\begin{array}{l}\alpha(gx,gu), \alpha(gy,gv), \alpha(gx,F(x,y)),\\ \alpha(gy,F(y,x)), \alpha(F(x,y),gu), \alpha(F(y,x),gv)\end{array}\right\}\right)$$

for all $x, y \in X, \phi \in \Phi$.

(6.5.6) (a) Assume F and g are continuous on X.

(or)

(6.5.6) (b) If $\{y_n\}$ is a sequence in X such that $\alpha(y_n, y_{n+1}) \geq d(y_n, y_{n+1})$ for $n \in \mathbb{N} \cup \{0\}$ and $y_n \to gy$ as $n \to \infty$ for some $y \in X$ then $\lim\limits_{n\to\infty} \alpha(y_n, gy) = 0$ and $\lim\limits_{n,\to\infty} \alpha(y_n, F(y,y)) \geq d(gy, F(y,y))$.

Then F and g have a unique common coupled fixed point.

Proof: From (6.5.4), ther exists $x_0 \in X$ such that

$$\alpha(gx_0, F(x_0,x_0)) \geq d(gx_0, F(x_0,x_0)). \tag{1}$$

From (6.5.1), there exists a sequence $\{x_n\}$ in X such that

$gx_{n+1} = F(x_n, x_n), n = 0, 1, 2, 3, ...$

From (1) $\alpha(gx_0, gx_1) \geq d(gx_0, gx_1)$.

From (6.5.3), $\alpha(F(x_0, x_0), F(x_1, x_1)) \geq d(F(x_0, x_0), F(x_1, x_1))$

$\Rightarrow \alpha(gx_1, gx_2) \geq d(gx_1, gx_2)$.

Again from (6.5.3), $\alpha(F(x_1, x_1), F(x_2, x_2)) \geq d(F(x_1, x_1), F(x_2, x_2))$

$\Rightarrow \alpha(gx_2, gx_3) \geq d(gx_2, gx_3)$.

Continuing in this way, we have

$$\alpha(gx_n, gx_{n+1}) \geq d(gx_n, gx_{n+1}) \ for \ n \ = \ 0, 1, 2, 3, ... \qquad (2)$$

Case(i): Suppose $gx_n = gx_{n+1}$ for some n.

Then $gx_n = F(x_n, x_n) \Rightarrow gz = F(z, z)$ where $z = x_n$.

Since the pair (F, g) is commuting, we have

$g^2z = ggz = g(F(z, z)) = F(gz, gz)$.

From (6.5.5), we have

$$\begin{aligned}\alpha(gz, g^2z) &= \alpha(F(z, z), F(gz, gz)) \\ &\leq \phi\left(\max\left\{\begin{array}{l}\alpha(gz, g^2z), \alpha(gz, g^2z), \alpha(gz, F(z, z)), \alpha(gz, F(z, z)), \\ \alpha(F(z, z), g^2z), \alpha(F(z, z), g^2z)\end{array}\right\}\right) \\ &= \phi(\alpha(gz, g^2z))\end{aligned}$$

which in turn yields that $\alpha(gz, g^2z) = 0$ since $\phi(t) < t$ for $t > 0$.

Thus $gz = g^2z$. Therfore $gz = g^2z = F(gz, gz)$.

Thus (gz, gz) is a common coupled fixed point of F and g.

Suppose (p, p) is a another common coupled fixed point of F and g.

i.e; $p = gp = F(p, p)$.

From (6.5.5), we have

$$\begin{aligned}\alpha(gz,p) &= \alpha(F(gz,gz),F(p,p)\\ &\leq \phi\left(\max\left\{\begin{array}{l}\alpha(g^2z,p),\alpha(g^2z,p),\alpha(g^2z,F(gz,gz)),\alpha(g^2z,F(gz,gz)),\\ \alpha(F(gz,gz),gp),\alpha(F(gz,gz),gp),\end{array}\right\}\right)\\ &= \phi(\alpha(gz,p))\end{aligned}$$

which in turn yields that $\alpha(gz,p)=0$ so that $gz=p$.

Thus (gz,gz) is the unique common coupled fixed point of F and g.

Case(ii):Assume that $gx_n \neq gx_{n+1}$ for all $n=0,1,2,....$

From (6.5.5), we have

$$\begin{aligned}\alpha(gx_n,gx_{n+1}) &= \alpha(F(x_{n-1},x_{n-1}),F(x_n,x_n))\\ &\leq \phi\left(\max\left\{\begin{array}{l}\alpha(gx_{n-1},gx_n),\alpha(gx_{n-1},gx_n),\alpha(gx_{n-1},gx_n),\\ \alpha(gx_{n-1},gx_n),\alpha(gx_n,gx_n),\alpha(gx_n,gx_n),\end{array}\right\}\right)\\ &= \phi(\alpha(gx_{n-1},gx_n)).\end{aligned}$$

From(2),

$$\begin{aligned}d(gx_n,gx_{n+1}) &\leq \alpha(gx_n,gx_{n+1})\\ &\leq \phi(\alpha(gx_{n-1},gx_n))\end{aligned}$$

continuing in this way and using the non-decreasing property of ϕ, we have

$$d(gx_n,gx_{n+1}) \leq \phi^n(\alpha(gx_0,gx_1)) \quad (3)$$

$$\rightarrow 0 \quad as \quad n \quad \rightarrow \quad \infty$$

Since $\sum_{n=1}^{\infty}\phi^n(t)<\infty$ for $t>0$, for each $\epsilon>0$ there exists a positive integer N such that $\sum_{n\geq N}\phi^n((\alpha(gx_0,gx_1))<\epsilon$.

Let m and n be positive integers such that $m>n\geq N$. Then
$d(gx_n,gx_m)\leq\sum_{k=n}^{m-1}d(gx_k,gx_{k+1})\leq\sum_{n\geq N}\phi^n((\alpha(gx_0,gx_1))<\epsilon.$

Thus $\{gx_n\}$ is Cauchy. Since $g(X)$ is complete, there exists $z \in X$ such that $gx_n \to gz$ as $n \to \infty$.

Suppose (6.5.6)(a) holds.

Then $g^2z = \underset{n\to\infty}{lim}\, ggx_{n+1} = \underset{n\to\infty}{lim}\, g(F(x_n, x_n)) = \underset{n\to\infty}{lim}\, F(gx_n, gx_n) = F(z, z)$.

Write $q = gz$. Then $gq = F(q, q)$.

Now as in Case(i), it follows that (gq, gq) is the unique common coupled fixed point of F and g.

Suppose (6.5.6)(b) holds.

Then from (6.5.6)(b), we have

$$\underset{n\to\infty}{lim}\, \alpha(gx_n, gz) = 0. \tag{4}$$

Also

$$\begin{aligned} d(gz, F(z, z)) &\leq \lim_{n\to\infty} \alpha(gx_n, F(z, z)) \\ &= \lim_{n\to\infty} \alpha(F(x_{n-1}, x_{n-1}), F(z, z)) \\ &\leq \lim_{n\to\infty} \phi\left(\max\left\{\begin{array}{l} \alpha(gx_{n-1}, gz), \alpha(gx_{n-1}, gz), \alpha(gx_{n-1}, gx_n), \\ \alpha(gx_{n-1}, gx_n), \alpha(gx_n, gz), \alpha(gx_n, gz) \end{array}\right\}\right) \\ &= \phi(0) \quad from (4) \quad and \quad continuity \quad of \quad \phi \end{aligned}$$

which in turn yields that $d(gz, F(z, z)) = 0$ so that

$$F(z, z) = gz. \tag{5}$$

Now as in Case(i) it follows that (gz, gz) is the unique common coupled fixed point of F and g.

Corollary 6.6. Let (X, d) be a metric space, $g : X \to X$ and $F : X \times X \to X$ be mappings. Let $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping with $\alpha(x, y) = 0 \Rightarrow x = y$. Assume (6.5.1),(6.5.2),(6.5.3),(6.5.4) and (6.4.6). Also assume

(6.6.1) $\alpha(F(x,y),F(u,v)) \le \phi(max\{\alpha(gx,gu),\alpha(gy,gv)\})$

for all $x,y,u,v \in X$ and $\phi \in \Phi$.

Then F and g have a unique common coupled fixed point .

Corollary 6.7. Let (X,d) be a complete metric space and $F : X \times X \to X$ be a mapping. Let $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping. Assume that

(6.7.1) $\alpha(F(x,y),F(u,v)) \le \phi(max\{\alpha(x,u),\alpha(y,v)\})$ for all $x,y,u,v \in X$ and $\phi \in \Phi$,

(6.7.2) F is α-admissible mapping with respect to d,

(6.7.3) there exists $x_0 \in X$ such that $\alpha(x_0, F(x_0,x_0) \ge d(x_0,F(x_0,x_0))$,

(6.7.4) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \ge d(x_n,x_{n+1})$ for all $n = 1,2,3...$ and $x_n \to x$ as $n \to \infty$, then $\lim_{n\to\infty} \alpha(x_n,x) = 0$ and $\lim_{n\to\infty} \alpha(x_n,F(x,x)) \ge d(x,F(x,x))$.

or

F is continuous on $X \times X$.

Then F has a coupled fixed point.

In 2016 Hussain et al.[69] proved the following theorem.

Theorem 6.8(Hussain et al.[69]): Let $\alpha : X \times X \to \mathbb{R}^+$ be a mapping (X,d) be a metric space and $T : X \to X$ be mapping and the following assertions hold:

(6.8.1) T is triangular α-admissible mapping with respect to $d(x,y)$,

(6.8.2) either T is continuous or,

(6.8.3) if $\{x_n\}$ is asequence in X such that $\alpha(x_n, x_{n+1}) \geq d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then $\lim_{n\to\infty} \alpha(x_n, Tx) \geq d(x, Tx)$,

(6.8.4) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq d(x_0, Tx_0)$,

(6.8.5) assume that there exists a function $\beta : \mathbb{R}^+ \to [0, 1]$ such that for any bounded sequence $\{t_n\}$ of posiytive reals ,$\beta(t_n) \to 1$ implies $t_n \to 0$ and for all $x,\ y\ \in X,\ \alpha(Tx, Ty) \leq \beta(\alpha(x, y))d(x, y)$.

Then T has a fixed point.

Definition 6.9. Let (X, d) be a metric space, $\alpha : X \times X \to \mathbb{R}^+$ be a mapping and $f, g, S, T : X \to X$. The pair (f, g) is α-admissible with respect to the pair (S, T) under d if for $x, y \in X,\ \alpha(Sx, Ty) \geq d(Sx, Ty) \Rightarrow \alpha(fx, gy) \geq d(fx, gy)$ and $\alpha(Tx, Sy) \geq d(Tx, Sy) \Rightarrow \alpha(gx, fy) \geq d(gx, fy)$.

Definition 6.10. (f, g) is called triangular α-admissible w.r.to the pair (S, T) if

(i) (f, g) is α-admissible w.r.to (S, T) and

(ii) $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$ for all $x, y, z \in X$.

Now we generalize the Hussain et al.[72] Theorem 6.7 for four maps as follows.

Theorem 6.11. Let $\alpha : X \times X \to \mathbb{R}^+$ be a mapping with $\alpha(x, y) = 0 \Rightarrow x = y$. Let (X, d) be a metric space and $f, g, S, T : X \to X$ be mappings satisfying

(6.11.1) $f(X) \subseteq T(X), g(X) \subseteq S(X)$,

(6.11.2) (f, S) and (g, T) are weakly compatible pairs, (6.11.3) $\alpha(fx, gy) \leq \beta(\alpha(Sx, Ty))d(Sx, Ty)$, for all $x, y \in X$, where $\beta : \mathbb{R}^+ \to [0, 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ as $n \to \infty$,

(6.11.4) there exists $x_1 \in X$ such that $\alpha(Sx_1, fx_1) \geq d(Sx_1, fx_1)$ and $\alpha(fx_1, Sx_1) \geq d(fx_1, Sx_1)$,

(6.11.5) the pair (f, g) is triangular α-admissible with respect to the pair (S, T) under d,

(6.11.6) suppose $S(X)$ is a complete sub space of X and $\lim_{n\to\infty} \alpha(fu, y_n) \geq d(fu, Su)$, $\lim_{n\to\infty} \alpha(Sz, y_n) \geq d(Sz, z)$ and $\alpha(z, Tz) \geq d(z, Tz)$ whenever there exists a sequence $\{y_n\}$ in X such that $\alpha(y_n, y_{n+1}) \geq d(y_n, y_{n+1})$ for $n = 1, 2, ...$ and $y_n \to z = Su$ for some $z, u \in X$.

Then f, g, S and T have a common fixed point.

Proof: From (6.11.4), there exists $x_1 \in X$ such that

$\alpha(Sx_1, fx_1) \geq d(Sx_1, fx_1)$ (1)

and $\alpha(fx_1, Sx_1) \geq d(fx_1, Sx_1)$ (2).

From (6.11.1), there exist sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$y_{2n+1} = fx_{2n+1} = Tx_{2n+2}$,

$y_{2n+2} = gx_{2n+2} = Sx_{2n+3}$, $n = 0, 1, 2, ...$

From(1),

$\alpha(Sx_1, fx_1) \geq d(Sx_1, fx_1)$

$\Rightarrow \alpha(Sx_1, Tx_2) \geq d(Sx_1, Tx_2)$ from definition of $\{y_n\}$

$\Rightarrow \alpha(fx_1, gx_2) \geq d(fx_1, gx_2)$ from (6.11.5), i.e; $\alpha(y_1, y_2) \geq d(y_1, y_2)$

$\Rightarrow \alpha(Tx_2, Sx_3) \geq d(Tx_2, Sx_3)$ from definition of $\{y_n\}$

$\Rightarrow \alpha(gx_2, fx_3) \geq d(gx_2, fx_3)$ from (6.11.5), i.e; $\alpha(y_2, y_3) \geq d(y_2, y_3)$

Continuing in this way, we have

$\alpha(y_n, y_{n+1}) \geq d(y_n, y_{n+1})$ for $n = 1, 2, ...$ (3)

Similarly using (2), we can show that

$\alpha(y_{n+1}, y_n) \geq d(y_{n+1}, y_n)$ for $n = 1, 2, ...$ (4)

By (3),(4) and using triangular property, we have

$\alpha(y_m, y_n) \geq d(y_m, y_n)$ for $m < n$ (5)

and $\alpha(y_n, y_m) \geq d(y_n, y_m)$ for $m < n$. (6)

Case(i): Suppose $y_{2m} = y_{2m+1}$ for some m.

Now from (3),

$$
\begin{aligned}
d(y_{2m+1}, y_{2m+2}) &\leq \alpha(y_{2m+1}, y_{2m+2}) \\
&= \alpha(fx_{2m+1}, gx_{2m+2}) \\
&\leq \beta(\alpha(Sx_{2m+1}, Tx_{2m+2}))d(Sx_{2m+1}, Tx_{2m+2}) \\
&= \beta(\alpha(y_{2m+1}, y_{2m+2}))(0) \\
&= 0
\end{aligned}
$$

which in turn yields that $y_{2m+1} = y_{2m+2}$.

Continuing in this way, we can show that $y_{2m} = y_{2m+1} = y_{2m+2} = ...$

Hence $\{y_n\}$ is a costant Cauchy sequence.

Case(ii): Suppose $y_n \neq y_{n+1}$ for all n.

As in case(i), $d(y_{2n+1}, y_{2n+2}) \leq \beta(\alpha(y_{2n}, y_{2n+1}))d(y_{2n}, y_{2n+1})$ (7)

Since $\beta(t) < 1$ and $y_n \neq y_{n+1}$ for all n, it follows that

$d(y_{2n+1}, y_{2n+2}) < d(y_{2n}, y_{2n+1})$. (8)

Consider

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(y_{2n+1}, y_{2n}) \\ &\leq \alpha(y_{2n+1}, y_{2n}) \quad from(4) \\ &= \alpha(fx_{2n+1}, gx_{2n}) \\ &\leq \beta(\alpha(Sx_{2n+1}, Tx_{2n}))d(Sx_{2n+1}, Tx_{2n}) \\ &= \beta(\alpha(y_{2n}, y_{2n-1}))d(y_{2n}, y_{2n-1}) \\ &< d(y_{2n}, y_{2n-1}). \end{aligned} \tag{9}$$

From (8) and (9), it follows that $\{d(y_n, y_{n+1})\}$ is a decreasing sequence of non-negative real numbers and hence converges to some real number $r \geq 0$ such that $\lim_{n\to\infty} d(y_n, y_{n+1}) = r$.

From (7), we have $\frac{d(y_{2n+1}, y_{2n+2})}{d(y_{2n}, y_{2n+1})} \leq \beta(\alpha(y_{2n}, y_{2n+1}))$.

Letting $n \to \infty$, we get $1 \leq \lim_{n\to\infty} \beta(\alpha(y_{2n}, y_{2n+1})) \leq 1$.

Hence $\lim_{n\to\infty} \alpha(y_{2n}, y_{2n+1}) = 0$.

But from (3), we have

$0 \leq \lim_{n\to\infty} d(y_{2n}, y_{2n+1}) \leq \lim_{n\to\infty} \alpha(y_{2n}, y_{2n+1}) = 0$.

Thus $\lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = 0$ and hence $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. (10)

Now we prove that $\{y_n\}$ is a Cauchy sequence. In view of (10), it sufficient to show that $\{y_{2n}\}$ is Cauchy.

Assume on the contray that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two sequences $\{y_{2m_k}\}$ and $\{y_{2n_k}\}$ of $\{y_{2n}\}$ so that n_k is the smallest positive integer such that $2n_k > 2m_k > k$ with

$d(y_{2m_k}, y_{2n_k}) \geq \epsilon$. (11)

and $d(y_{2m_k}, y_{2n_k - 2}) < \epsilon$. (12)

From(11) and (12), we have

$$\begin{aligned}\epsilon \le d(y_{2m_k}, y_{2n_k}) &\le d(y_{2m_k}, y_{2n_k-2}) + d(y_{2n_k-2}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k}) \\ &< \epsilon + d(y_{2n_k-2}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k})\end{aligned}$$

letting $k \to \infty$ and using (10), we get $\epsilon \le \lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) \le \epsilon$ so that

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \epsilon \qquad (13)$$

letting $k \to \infty$ and using (10) and (13) in

$|d(y_{2m_k+1}, y_{2n_k}) - d(y_{2m_k}, y_{2n_k})| \le d(y_{2m_k}, y_{2m_k+1})$ we have

$$\lim_{k\to\infty} d(y_{2m_k+1}, y_{2n_k}) = \epsilon \qquad (14)$$

letting $k \to \infty$ and using (10) and (11) in

$|d(y_{2m_k}, y_{2n_k-1}) - d(y_{2m_k}, y_{2n_k})| \le d(y_{2n_k-1}, y_{2n_k})$,

we get

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k-1}) = \epsilon. \qquad (15)$$

From (5), we have

$$\begin{aligned}d(y_{2m_k+1}, y_{2n_k}) &\le \alpha(y_{2m_k+1}, y_{2n_k}) \\ &= \alpha(fx_{2m_k+1}, gx_{2n_k}) \\ &\le \beta(\alpha(y_{2m_k}, y_{2n_k-1}))d(y_{2m_k}, y_{2n_k-1}) \qquad (16)\end{aligned}$$

letting $k \to \infty$ in (16), we get

$\epsilon \le \lim_{k\to\infty} \beta(\alpha(y_{2m_k}, y_{2n_k-1}))\epsilon$ from (14),(15)

$1 \le \lim_{k\to\infty} \beta(\alpha(y_{2m_k}, y_{2n_k-1}))$.

But $\lim_{k\to\infty} \beta(\alpha(y_{2m_k}, y_{2n_k-1})) \le 1$.

Thus $\lim_{k\to\infty} \beta(\alpha(y_{2m_k}, y_{2n_k-1})) = 0$. Hence $\lim_{k\to\infty} \alpha(y_{2m_k}, y_{2n_k-1}) = 0$.

From (5) and (6) we have

$\underset{k\to\infty}{lim}\, d(y_{2m_k}, y_{2n_k-1}) = 0$

$\epsilon = 0$ from (15). It is a contradiction.

Hence $\{y_{2n}\}$ is a Cauchy sequence.

From (10), it follows that $\{y_{2n+1}\}$ is also a Cauchy sequence.

Thus $\{y_n\}$ is a Cauchy sequence.

Suppose (6.11.6) holds.

Since $y_{2n+2} = Sx_{2n+3} \in S(X)$ and $S(X)$ is a complete subspace of X, there exist z and $u \in X$ such that $y_{2n+2} \to z = Su$.

Thus $\underset{n\to\infty}{lim}\, gx_{2n+2} = \underset{n\to\infty}{lim}\, fx_{2n+1} = \underset{n\to\infty}{lim}\, Tx_{2n+2} = z$.

Now we have

$$\begin{aligned} d(fu, z) = d(fu, Su) &\leq \lim_{n\to\infty} \alpha(fu, y_{2n+2}) \\ &= \lim_{n\to\infty} \alpha(fu, gx_{2n+2}) \\ &\leq \lim_{n\to\infty} \beta(\alpha(Su, Tx_{2n+2}))d(Su, Tx_{2n+2}) \\ &= \lim_{n\to\infty} \beta(\alpha(z, y_{2n+1}))d(z, y_{2n+1}) = 0 \end{aligned}$$

which in turn yields that $Su = z = fu$.

Since (f, S) is weakly compatible, we have $fz = fSu = Sfu = Sz$

From (6.11.6), we have

$$\begin{aligned} d(Sz, z) &\leq \lim_{n\to\infty} \alpha(Sz, y_{2n+2}). \qquad (17) \\ &= \lim_{n\to\infty} \alpha(fz, gx_{2n+2}) \\ &\leq \lim_{n\to\infty} \beta(\alpha(Sz, Tx_{2n+2}))d(Sz, Tx_{2n+2}) \\ &= \lim_{n\to\infty} \beta(\alpha(Sz, y_{2n+1}))d(Sz, y_{2n+1}) \\ &= \lim_{n\to\infty} \beta(\alpha(Sz, y_{2n+1}))d(Sz, z) \end{aligned}$$

Thus $1 \leq \lim_{n\to\infty} \beta(\alpha(Sz, y_{2n+1})) \leq 1$

which in turn yields that $\lim_{n\to\infty} \alpha(Sz, y_{2n+1}) = 0$.

Hence from (17), $d(Sz, z) = 0$ so that $Sz = z$.

Thus $fz = Sz = z$.

Since $f(X) \subseteq T(X)$, there exist $\alpha \in X$ such that $z = fz = T\alpha$.

Now

$$\begin{aligned}\alpha(z, g\alpha) = \alpha(T\alpha, g\alpha) &= \alpha(fz, g\alpha)\\ &\leq \beta(\alpha(Sz, T\alpha))d(Sz, T\alpha)\\ &= \beta(\alpha(z, z))d(z, z)\\ &= 0\end{aligned}$$

which in turn yields that $\alpha(z, g\alpha) = 0$

Thus $g\alpha = z = T\alpha$

Since (g, T) is weakly compatible, we have $Tz = T(g\alpha) = gT\alpha = gz$.

Now

$$\begin{aligned}\alpha(z, Tz) &= \alpha(fz, gz)\\ &\leq \beta(\alpha(Sz, Tz))d(Sz, Tz)\\ &= \beta(\alpha(z, Tz))d(z, Tz).\end{aligned}$$

Since $\beta(t) < 1$ we have $\alpha(z, Tz) = 0$ so that $Tz = z$. Thus $Tz = gz = z$.

Thus z is a common fixed point of f, g, S and T.

CHAPTER 7

UNIQUE COMMON FIXED POINT THEOREM OF INTEGRAL TYPE FOR FOUR MAPS IN DISLOCATED QUASI b-METRIC SPACES

In this chapter we obtain two common fixed point theorems using contractive conditions of integral type in dislocated quasi b-metric spaces. We also furnish examples which supports our results.

In 2002, Banaciari et al.[6] generalized the Banach contraction principle by introducing the integral contraction. Afterwards many researchers extended the results of Banaciari and obtained fixed point and common fixe point theorems using various contractive conditions of integral type in different spaces for example (refer[1, 7, 25, 37, 65, 112]).

Before proving our theorems, we state the following.

Definition 7.1. Let Γ denote the class of functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ which are Lebesgue summable on each compact subset of $\mathbb{R}^+$, non-negative and $\int_0^s \rho(t)\ dt\ > 0$ for any $s > 0$. We observe the following

(i) $\lim\limits_{n\to\infty} \int_0^{a_n} \rho(t)\ dt = \int_0^{\lim\limits_{n\to\infty} a_n} \rho(t)\ dt$ for any non-negative real sequence $\{a_n\}$.

(ii) $max\{\int_0^a \rho(t)\ dt, \int_0^b \rho(t)\ dt\} = \int_0^{max\{a,b\}} \rho(t)\ dt$ for any non-negative real numbers a and b.

(iii) $\int_0^a \rho(t)\ dt\ \le h \int_0^a \rho(t)\ dt$ for any non-negative real number a and $0 \le h < 1$ implies $a = 0$.

Definition 7.2(M.U.Ali et al.[65]): $\rho \in \Gamma$ is an integral sub additive if for each $a, b > 0$ one has $\int_0^{a+b} \rho(t)dt \leq \int_0^a \rho(t)dt + \int_0^b \rho(t)dt$.

In the year 2018 M U Rahman et al.[67] proved the following theorem.

Theorem 7.3.(M U Rahman et al.[67]): Let (X, d) be a complete dislocated quasi b-metric space, for $a, b, c, e, f \geq 0$ with $\frac{a+b}{1-(c+e+f)} < \frac{1}{k}$, where $k \geq 1$ and let $T : X \to X$ be a continuous self-mapping such that for all $x, y \in X$ satisfying the condition

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leq a \int_0^{d(x,y)} \rho(t)dt + b \int_0^{d(x,Tx)} \rho(t)dt + c \int_0^{d(y,Ty)} \rho(t)dt + e \int_0^{\frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}} \rho(t)dt + f \int_0^{\frac{d(x,Ty)d(y,Ty)}{k[d(x,y)+d(y,Ty)]}} \rho(t)dt$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$, $\int_0^s \rho(t)dt > 0$. Then T has a unique fixed point.

In proving Theorem 7.3 the authors [67] inherently used integral sub additive definition of [65] (namely,Definition 7.2)

Now we prove two unique common fixed point theorems for four maps using integral type conditions in dislocated quasi b-metric spaces.

Theorem 7.4. Let (X, d) be a complete dislocated quasi b- metric space with fixed integer $k \geq 1$, $0 \leq h < 1$ with $hk < 1$ and $F, G, S, T : X \to X$ be continuous mapping satisfying

(7.4.1) $\int_0^{d(Fx,Gy)} \rho(t)dt \leq h \int_0^{M_1(x,y)} \rho(t)dt$, for all $x, y \in X$ and $\rho \in \Gamma$ where

$$M_1(x,y) = \max \left\{ \begin{array}{l} d(Sx,Ty), \frac{1}{2k}d(Sx,Fx), \frac{1}{2k}d(Ty,Gy) \\ \frac{1}{2k}d(Sx,Gy), \frac{1}{2k}d(Ty,Fx) \end{array} \right\}$$

(7.4.2) $\int_0^{d(Gx,Fy)} \rho(t)dt \leq h \int_0^{M_2(x,y)} \rho(t)dt$, for all $x, y \in X$ and $\rho \in \Gamma$ where

$$M_2(x,y) = \max \left\{ \begin{array}{l} d(Tx,Sy), \frac{1}{2k}d(Tx,Gx), \frac{1}{2k}d(Sy,Fy) \\ \frac{1}{2k}d(Tx,Fy), \frac{1}{2k}d(Sy,Gx) \end{array} \right\}$$

(7.4.3) $F(X) \subseteq T(X)$ and $G(X) \subseteq S(X)$,

(7.4.4) $FS = SF$ and $GT = TG$.

Then F, G, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$.

From (7.4.3), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$y_{2n} = Fx_{2n} = Tx_{2n+1}$,

$y_{2n+1} = Gx_{2n+1} = Sx_{2n+2}, n = 0, 1, 2, ...$

Case(i): Suppose $max\{d(y_{n-1}, y_n), d(y_n, y_{n-1})\} = 0$ for some n.

With out loss of generality assume that $n = 2m$. Then $y_{2m-1} = y_{2m}$.

Consider from (7.4.1),

$$\int_0^{d(y_{2m},y_{2m+1})} \rho(t)dt = \int_0^{d(Fx_{2m},Gx_{2m+1})} \rho(t)dt \leq h \int_0^{d(x_{2m},x_{2m+1})} \rho(t)dt$$

From Note 1.12.6 (Ch-1) and Case(i), we have

$$M_1(x_{2m}, x_{2m+1}) = \max \left\{ \begin{array}{l} d(y_{2m-1}, y_{2m}), \frac{1}{2k}d(y_{2m-1}, y_{2m}), \frac{1}{2k}d(y_{2m}, y_{2m+1}), \\ \frac{1}{2k}d(y_{2m-1}, y_{2m+1}), \frac{1}{2k}d(y_{2m}, y_{2m}), \end{array} \right\}$$
$$\leq \max \left\{ \begin{array}{l} d(y_{2m-1}, y_{2m}), d(y_{2m-1}, y_{2m}), d(y_{2m}, y_{2m+1}), \\ \max\{d(y_{2m-1}, y_{2m}), d(y_{2m}, y_{2m+1})\}, \\ \max\{d(y_{2m}, y_{2m+1}), d(y_{2m+1}, y_{2m})\} \end{array} \right\}$$
$$= \max\{d(y_{2m}, y_{2m+1}), d(y_{2m+1}, y_{2m})\}.$$

Thus we have $\int_0^{d(y_{2m},y_{2m+1})} \rho(t)dt \ \leq \ h \int_0^{max\{d(y_{2m},y_{2m+1}),d(y_{2m+1},y_{2m})\}} \rho(t)dt$

From (7.4.2), we have

$$\int_0^{d(y_{2m+1},y_{2m})} \rho(t)dt = \int_0^{d(Gx_{2m+1},Fx_{2m})} \rho(t)dt$$
$$\leq h \int_0^{M_2(x_{2m+1},x_{2m})} \rho(t)dt$$

From Note 1.12.6 (Ch-1) and Case(i), we have

$$M_2(x_{2m+1}, x_{2m}) = \max \left\{ \begin{array}{l} d(y_{2m}, y_{2m-1}), \frac{1}{2k}d(y_{2m}, y_{2m+1}), \frac{1}{2k}d(y_{2m-1}, y_{2m}), \\ \frac{1}{2k}d(y_{2m}, y_{2m}), \frac{1}{2k}d(y_{2m-1}, y_{2m+1}) \end{array} \right\}$$
$$\leq \max \left\{ \begin{array}{l} d(y_{2m}, y_{2m-1}), d(y_{2m}, y_{2m+1}), d(y_{2m-1}, y_{2m}), \\ \max \{d(y_{2m}, y_{2m+1}), d(y_{2m+1}, y_{2m})\}, \\ \max \{d(y_{2m-1}, y_{2m}), d(y_{2m}, y_{2m+1})\} \end{array} \right\}$$
$$= \max \{d(y_{2m}, y_{2m+1}), d(y_{2m+1}, y_{2m})\}.$$

Thus

$$\int_0^{d(y_{2m+1},y_{2m})} \rho(t)dt \ \leq \ h \int_0^{max\{d(y_{2m},y_{2m+1}),d(y_{2m+1},y_{2m})\}} \rho(t)dt.$$

Hence we have

$$\int_0^{max\{d(y_{2m},y_{2m+1}),d(y_{2m+1},y_{2m})\}} \rho(t)dt \ \leq \ h \int_0^{max\{d(y_{2m},y_{2m+1}),d(y_{2m+1},y_{2m})\}} \rho(t)dt$$

which in turn yields that $max\{d(y_{2m}, y_{2m+1}), d(y_{2m}, y_{2m+1})\} = 0$

so that, $y_{2m} = y_{2m+1}$.

Continuing in this way, we can show that $y_{2m-1} = y_{2m} = y_{2m+1} = ...$

Thus $\{y_n\}$ is a constant Cauchy sequence in X.

Case(ii): Suppose that $max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\} > 0$ for all n.

As in Case(i), we have

$$\int_0^{\max\{d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n})\}} \rho(t)dt \ \leq \ h \int_0^{\max\left\{ \begin{array}{l} d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n-1}), \\ d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n}) \end{array} \right\}} \rho(t)dt \quad (1)$$

If $max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1})\} \ \leq \ max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\}$

then from (1), we have

$$\int_0^{\max\{d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n})\}} \rho(t)dt \leq h \int_0^{\max\{d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n})\}} \rho(t)dt$$

which in turn yields that $max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\} = 0.$

It is a contradiction to the Case(ii).

Hence from (1), we have

$$\int_0^{\max\{d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n})\}} \rho(t)dt \leq h \int_0^{\max\{d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n-1})\}} \rho(t)dt.$$

$$\begin{aligned} \int_0^{\max\{d(y_n,y_{n+1}),d(y_{n+1},y_n)\}} \rho(t)dt &\leq h \int_0^{\max\{d(y_{n-1},y_n),d(y_n,y_{n-1})\}} \rho(t)dt \\ &\vdots \\ &\vdots \\ &\leq h^n \int_0^{\max\{d(y_0,y_1),d(y_1,y_0)\}} \rho(t)dt \end{aligned} \tag{2}$$

Now for all positive integers $n,\ p$ and using(2), we have

$d(y_n, y_{n+p}) \leq kd(y_n, y_{n+1}) + k^2 d(y_{n+1}, y_{n+2}) + ... + k^p d(y_{n+p-1}, y_{n+p})$

Since ρ is integral sub additive, we have

$$\int_0^{d(y_n,y_{n+p})} \rho(t)dt$$
$$\leq \int_0^{kd(y_n,y_{n+1})+k^2 d(y_{n+1},y_{n+2})+...+k^p d(y_{n+p-1},y_{n+p})} \rho(t)dt$$
$$= \int_0^{kd(y_n,y_{n+1})} \rho(t)dt + \int_0^{k^2 d(y_{n+1},y_{n+2})} \rho(t)dt + ... + \int_0^{k^p d(y_{n+p-1},y_{n+p})} \rho(t)dt$$
$$\leq k \int_0^{d(y_n,y_{n+1})} \rho(t)dt + k^2 \int_0^{d(y_{n+1},y_{n+2})} \rho(t)dt + ... + k^p \int_0^{d(y_{n+p-1},y_{n+p})} \rho(t)dt$$
$$\leq kh^n \int_0^{\max\{d(y_0,y_1),d(y_1,y_0)\}} \rho(t)dt + k^2 h^{n+1} \int_0^{\max\{d(y_0,y_1),d(y_1,y_0)\}} \rho(t)dt + ...$$
$$+ k^p h^{n+p-1} \int_0^{\max\{d(y_0,y_1),d(y_1,y_0)\}} \rho(t)dt$$
$$\leq \frac{kh^n}{1-kh} \int_0^{\max\{d(y_0,y_1),d(y_1,y_0)\}} \rho(t)dt \ since\ hk < 1$$
$$\to 0 \ as\ n \to \infty, \ since\ 0 \leq h < 1.$$

Thus $\lim_{n\to\infty} d(y_n, y_{n+p}) = 0$.

Similarly we can show that $\lim_{n\to\infty} d(y_{n+p}, y_n) = 0$.

Thus $\{y_n\}$ is a Cauchy sequence in X.

Since X is complete dislocated quasi b-metric space, there exists $z \in X$ such that $\{y_n\}$ converges to z. Since S and F are continuous and $SF = FS$ we have $Sz = \lim_{n\to\infty} Sy_{2n} = \lim_{n\to\infty} SFx_{2n} = \lim_{n\to\infty} FSx_{2n} = \lim_{n\to\infty} Fy_{2n-1} = Fz$.

Similarly, since T and G are continuous and $TG = GT$ we have $Tz = Gz$.

From (7.4.1), we have

$$\int_0^{d(Sz,Tz)} \rho(t)dt = \int_0^{d(Fz,Gz)} \rho(t)dt \ \leq \ \int_0^{M_1(z,z)} \rho(t)dt$$

$$M_1(z,z) = \max\left\{ \begin{array}{l} d(Sz,Tz), \frac{1}{2k}d(Sz,Fz), \frac{1}{2k}d(Tz,Gz), \\ \frac{1}{2k}d(Sz,Gz), \frac{1}{2k}d(Tz,Fz) \end{array} \right\}$$

$$\leq \max\left\{ \begin{array}{l} d(Sz,Tz), \max\{d(Sz,Tz), d(Tz,Sz)\}, \\ \max\{d(Tz,Sz), d(Sz,Tz)\}, d(Sz,Tz), d(Tz,Sz) \end{array} \right\}$$

$$= \max\{d(Sz,Tz), d(Tz,Sz)\}.$$

Thus $\int_0^{d(Sz,Tz)} \rho(t)dt \ \leq \ h \int_0^{max\{d(Sz,Tz),d(Tz,Sz)\}} \rho(t)dt.$

Similarly using (7.4.2), we can show that

$\int_0^{d(Tz,Sz)} \rho(t)dt \ \leq \ h \int_0^{max\{d(Sz,Tz),d(Tz,Sz)\}} \rho(t)dt.$

Thus we have

$\int_0^{max\{d(Sz,Tz),d(Tz,Sz)\}} \rho(t)dt \ \leq \ h \int_0^{max\{d(Sz,Tz),d(Tz,Sz)\}} \rho(t)dt$

which in turn yields that $max\{d(Sz,Tz), d(Tz,Sz)\} = 0$.

Hence $Sz = Tz$.

Let $u = Sz = Tz$.

Then $Su = S(Sz) = S(Fz) = F(Sz) = Fu$ and

$Tu = T(Tz) = T(Gz) = G(Tz) = Gu$.

From (7.4.1), we have

$$\int_0^{d(Su,u)} \rho(t)dt = \int_0^{d(Fu,Gz)} \rho(t)dt$$
$$\leq h \int_0^{M_1(u,z)} \rho(t)dt$$

$$M_1(u,z) = \max\left\{\begin{array}{l} d(Su,Tz), \frac{1}{2k}d(Su,Fu), \frac{1}{2k}d(Tz,Gz), \\ \frac{1}{2k}d(Su,Gz), \frac{1}{2k}d(Tz,Fu) \end{array}\right\}$$
$$\leq \max\left\{\begin{array}{l} d(Su,u), \max\{d(Su,u), d(u,Su)\}, \\ \max\{d(u,Su), d(Su,u)\}, d(Su,u), d(u,Su) \end{array}\right\}$$
$$= \max\{d(Su,u), d(u,Su)\}.$$

Thus $\int\limits_0^{d(Su,u)} \rho(t)dt \leq h \int\limits_0^{max\{d(Su,u),d(u,Su)\}} \rho(t)dt.$

Similarly, using (7.4.2), we can show that
$\int\limits_0^{d(u,Su)} \rho(t)dt \leq h \int\limits_0^{max\{d(Su,u),d(u,Su)\}} \rho(t)dt.$

Thus we have
$\int\limits_0^{max\{d(Su,u),d(u,Su)\}} \rho(t)dt \leq h \int\limits_0^{max\{d(Su,u),d(u,Su)\}} \rho(t)dt$

which in turn yields that $max\{d(Su,u), d(u,Su)\} = 0$.

Hence $Su = u$.

Thus $Su = u = Fu$.

Similarly we can show that $Tu = u = Gu$.

Thus u is a common fixed point of F, G, S and T.

Let u^* be another common fixed point of F, G, S and T.

Then $Fu^* = Su^* = u^* = Tu^* = Gu^*$.

Consider from (7.4.1), we have

$$\int\limits_0^{d(u,u^*)} \rho(t)dt = \int\limits_0^{d(Fu,Gu^*)} \rho(t)dt$$
$$\leq h \int\limits_0^{M_1(u,u^*)} \rho(t)dt$$

$$M_1(u,u^*) = \max\left\{\begin{array}{l} d(Su,Tu^*), \frac{1}{2k}d(Su,Fu), \frac{1}{2k}d(Tu^*,Gu^*), \\ \frac{1}{2k}d(Su,Gu^*), \frac{1}{2k}d(Tu^*,Fu) \end{array}\right\}$$
$$\leq \max\left\{\begin{array}{l} d(u,u^*), \max\{d(u,u^*),d(u^*,u)\}, \\ \max\{d(u^*,u),d(u,u^*)\}, d(u,u^*), d(u^*,u) \end{array}\right\}$$
$$= \max\{d(u,u^*), d(u^*,u)\}.$$

Thus $\int_0^{d(u,u^*)} \rho(t)dt \leq h \int_0^{max\{d(u,u^*),d(u^*,u)\}} \rho(t)dt.$

Similarly using (7.4.2), we can show that

$\int_0^{d(u^*,u)} \rho(t)dt \leq h \int_0^{max\{d(u,u^*),d(u^*,u)\}} \rho(t)dt.$

Thus we have

$\int_0^{max\{d(u,u^*),d(u^*,u)\}} \rho(t)dt \leq h \int_0^{max\{d(u,u^*),d(u^*,u)\}} \rho(t)dt$

which in turn yields that $max\{d(u,u^*),d(u^*,u)\} = 0$

Hence $u = u^*$.

Thus u is unique common fixed point of F, G, S and T.

Now we give an example to illustrate our Theorem 7.4

Example 7.5. Let $X = [0,1]$ and $d(x,y) = |2x-y|^2 + |2x+y|^2$.

Let $F, G, S, T : X \to X$ be defined by $Fx = \frac{x}{16}, Gx = \frac{x}{24}$,

$Sx = \frac{x}{4}$ and $Tx = \frac{x}{6}$.

Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\rho(t) = 1$.

Clearly $d(x,y) = d(y,x) = 0$ implies that $x = y$.

Consider

$d(x,y) = |2x-y|^2 + |2x+y|^2$

$$\begin{aligned}
d(x,y) &= |2x-y|^2 + |2x+y|^2 \\
&= |2x-z+z-y|^2 + |2x+z-z+y|^2 \\
&\le 2\left[|2x-z|^2 + |z-y|^2 + |2x+z|^2 + |-z+y|^2\right] \\
&\le 2\left[|2x-z|^2 + |2z-y|^2 + |2x+z|^2 + |2z+y|^2\right] \\
&= 2\left[|2x-z|^2 + |2x+z|^2 + |2z-y|^2 + |2z+y|^2\right] \\
&= k\left[d(x,z) + d(z,y)\right], \ \textit{where } k = 2
\end{aligned}$$

Consider

$$\begin{aligned}
\int_0^{d(Fx,Gy)} \rho(t)dt &= \int_0^{d(\frac{x}{16},\frac{y}{24})} 1 \ dt \\
&= [t]_0^{d(\frac{x}{16},\frac{y}{24})} \\
&= d(\tfrac{x}{16},\tfrac{y}{24}) \\
&= \left|\tfrac{2x}{16} - \tfrac{y}{24}\right|^2 + \left|\tfrac{2x}{16} + \tfrac{y}{24}\right|^2 \\
&= \tfrac{1}{16}\left[\left|\tfrac{2x}{4} - \tfrac{y}{6}\right|^2 + \left|\tfrac{2x}{4} + \tfrac{y}{6}\right|^2\right] \\
&= \tfrac{1}{16}\left[\left|\tfrac{2x}{4} - \tfrac{y}{6}\right|^2 + \left|\tfrac{2x}{4} + \tfrac{y}{6}\right|^2\right] \\
&= \tfrac{1}{16}\left[d(Sx,Ty)\right] \\
&= \tfrac{1}{16}\int_0^{d(Sx,Ty)} 1 \ dt \\
&\le h \int_0^{M_1(x,y)} \rho(t) \ dt
\end{aligned}$$

$$\textit{where } M_1(x,y) = \max\left\{\begin{array}{l} d(Sx,Ty), \frac{1}{2k}d(Sx,Fx), \frac{1}{2k}d(Ty,Gy), \\ \frac{1}{2k}d(Sx,Gy), \frac{1}{2k}d(Ty,Fx) \end{array}\right\}$$

Thus (7.4.1) is satisfied. Similarly we can verify (7.4.2). Also it is clear that F, G, S and T are continuous, $FS = SF$, $GT = TG$ and $F(x) \subseteq T(X), G(X) \subseteq S(X)$. Thus all conditions of Theorem 7.4 are satisfied. Clearly 0 is the unique common fixed point of F, G, S and T in X.

Theorem 7.6. Let (X, d) be a dislocated quasi b-metric space with fixed integer $k \geq 1, 0 \leq h < 1$ and $F, G, S, T : X \to X$ be mappings satisfying

(7.6.1) $\int_0^{d(Fx,Gy)} \rho(t)dt \leq h \int_0^{M_1(x,y)} \rho(t)dt$ for all $x, y \in X$, $\rho \in \Gamma$ and ρ is integral sub additive and

$M_1(x, y) = \frac{1}{2k^2}\ max\ \{d(Sx, Ty), d(Sx, Fx), d(Ty, Gy), d(Sx, Gy), d(Ty, Fx)\}$

(7.6.2) $\int_0^{d(Gx,Fy)} \rho(t)dt \leq h \int_0^{M_2(x,y)} \rho(t)dt$ for all $x, y \in X$ $\rho \in \Gamma$ and ρ is integral sub additive and

$M_2(x, y) = \frac{1}{2k^2}\ max\ \{d(Tx, Sy), d(Tx, Gx), d(Sy, Fy), d(Tx, Fy), d(Sy, Gx)\}$

(7.6.3) $F(X) \subseteq T(X)$ and $G(X) \subseteq S(X)$,

(7.6.4) one of $T(X)$ and $S(X)$ is a complete subspace of X and

(7.6.5) the pairs (F, S) and (G, T) are weakly compatible.

Then F, G, S and T have a unique common fixed point in X.

Proof: As in proof of Theorem 7.4, the sequence$\{y_n\}$ is Cauchy

where $y_{2n} = Fx_{2n} = Tx_{2n+1}$,

$y_{2n+1} = Gx_{2n+1} = Sx_{2n+2}, n = 0, 1, 2, ...$

Assume that $S(X)$ is a complete subspace of X.

Since $y_{2n+1} = Sx_{2n+2} \in S(X)$, there exists $z \in S(X)$ such that $y_{2n+1} \to z$.

Hence there exists $u \in X$ such that $z = Su$.

Since $\{y_n\}$ is Cauchy sequence we have $y_{2n} \to z$.

By Lemma 1.12.5 (Ch-1) and (7.6.1) we have

$$\int_0^{\frac{1}{k}d(Fu,z)} \rho(t)dt \leq \int_0^{\lim\limits_{n\to\infty} d(Fu,Gx_{2n+1})} \rho(t)dt$$

$$= \lim_{n\to\infty} \int_0^{d(Fu,Gx_{2n+1})} \rho(t)dt$$

$$\leq h \lim_{n\to\infty} \int_0^{M_1(u,x_{2n+1})} \rho(t)dt$$

$$\lim_{n\to\infty} M_1(u, x_{2n+1}) = \lim_{n\to\infty} \tfrac{1}{2k^2} \max \left\{ \begin{array}{l} d(Su, Tx_{2n+1}), d(Su, Fu), d(Tx_{2n+1}, Gx_{2n+1}), \\ d(Su, Gx_{2n+1}), d(Tx_{2n+1}, Fu) \end{array} \right\}$$

$$\leq \lim_{n\to\infty} \tfrac{1}{2k^2} \max \left\{ \begin{array}{l} d(z, y_{2n}), d(z, Fu), d(y_{2n}, y_{2n+1}), \\ d(z, y_{2n+1}), d(y_{2n}, Fu) \end{array} \right\}$$

$$\leq \tfrac{1}{2k} d(z, Fu), \ \ by \ Lemma(1.12.5)(Ch-1)$$

$$\leq \tfrac{1}{k} d(z, Fu).$$

Thus

$$\int_0^{\frac{1}{k}d(Fu,z)} \rho(t)dt \leq h \int_0^{\frac{1}{k}d(z,Fu)} \rho(t)dt \qquad (1)$$

By Lemma 1.12.5 and (7.6.2), we have

$$\int_0^{\frac{1}{k}d(z,Fu)} \rho(t)dt \leq \int_0^{\lim_{n\to\infty} d(Gx_{2n+1},Fu)} \rho(t)dt$$

$$= \lim_{n\to\infty} \int_0^{d(Gx_{2n+1},Fu)} \rho(t)dt$$

$$\leq h \lim_{n\to\infty} \int_0^{M_2(x_{2n+1},u)} \rho(t)dt$$

$$\lim_{n\to\infty} M_1(x_{2n+1}, Fu) = \lim_{n\to\infty} \tfrac{1}{2k^2} \max \left\{ \begin{array}{l} d(Tx_{2n+1}, Su), d(Tx_{2n+1}, Gx_{2n+1}), d(Su, Fu), \\ , d(Tx_{2n+1}, Fu), d(Su, Gx_{2n+1}) \end{array} \right\}$$

$$\leq \lim_{n\to\infty} \tfrac{1}{2k^2} \max \left\{ \begin{array}{l} d(y_{2n}, z), d(y_{2n}, y_{2n+1}), d(z, Fu), \\ d(y_{2n}, Fu), d(z, y_{2n+1}) \end{array} \right\}$$

$$\leq \tfrac{1}{2k} d(z, Fu), by \ \ Lemma \ 1.12.5 \ (Ch-1)$$

$$\leq \tfrac{1}{k} d(z, Fu).$$

Thus we have

$$\int_0^{\frac{1}{k}d(z,Fu)} \rho(t)dt \leq h \int_0^{\frac{1}{k}d(z,Fu)} \rho(t)dt$$

which in turn yields that $d(z, Fu) = 0$

From(1), $d(Fu, z) = 0$. Thus $z = Fu$.

Hence $Su = z = Fu$.

Since (F, S) is weakly compatible, we have $Sz = SFu = FSu = Fz$.

From Lemma 1.12.5 (Ch-1) and using (7.6.1), we have

$$\int_0^{\frac{1}{k}d(Sz,z)} \rho(t)dt \leq \int_0^{\lim_{n\to\infty} d(Fz,Gx_{2n+1})} \rho(t)dt$$
$$= \lim_{n\to\infty} \int_0^{d(Fz,Gx_{2n+1})} \rho(t)dt$$
$$\leq h \lim_{n\to\infty} \int_0^{M_1(z,x_{2n+1})} \rho(t)dt$$

$$\lim_{n\to\infty} M_1(z, x_{2n+1}) = \lim_{n\to\infty} \frac{1}{2k^2} \max \left\{ \begin{array}{l} d(Sz, Tx_{2n+1}), d(Sz, Fz), d(Tx_{2n+1}, Gx_{2n+1}), \\ d(Sz, Gx_{2n+1}), d(Tx_{2n+1}, Fz) \end{array} \right\}$$
$$\leq \lim_{n\to\infty} \frac{1}{2k^2} \max \left\{ \begin{array}{l} d(Sz, y_{2n}), d(z, Fz), d(y_{2n}, y_{2n+1}), \\ d(Sz, y_{2n+1}), d(y_{2n}, Fz) \end{array} \right\}$$
$$\leq \frac{1}{2k} \max \{d(Sz, z), d(z, Sz)\}$$
$$\leq \frac{1}{k} \max \{d(Sz, z), d(z, Sz)\}.$$

Thus

$$\int_0^{\frac{1}{k}d(Sz,z)} \rho(t)dt \leq h \int_0^{\frac{1}{k}\max\{d(Sz,z),d(z,Sz)\}} \rho(t)dt.$$

Similarly using (7.6.2) and Lemma 1.12.5 (Ch-1) we can show that

$$\int_0^{\frac{1}{k}d(z,Sz)} \rho(t)dt \leq h \int_0^{\frac{1}{k}\max\{d(Sz,z),d(z,Sz)\}} \rho(t)dt.$$

Thus we have

$$\int_0^{\frac{1}{k}\max\{d(Sz,z),d(z,Sz)\}} \rho(t)dt \leq h \int_0^{\frac{1}{k}\max\{d(Sz,z),d(z,Sz)\}} \rho(t)dt$$

which in turn yields that $Sz = z$.

Thus $Fz = Sz = z$.

Since $F(X) \subseteq T(X)$, there exists $v \in X$ such that $z = Sz = Fz = Tv$

Consider from (7.6.1), we have

$$\int_0^{d(Tv,Gv)} \rho(t)dt = \int_0^{d(Fz,Gv)} \rho(t)dt$$
$$\leq h \int_0^{M_1(z,v)} \rho(t)dt$$

$$M_1(z,v) = \frac{1}{2k^2} \max \left\{ \begin{array}{l} d(Sz,Tv), d(Sz,Fz), d(Tv,Gv), \\ d(Sz,Gv), d(Tv,Fz) \end{array} \right\}$$
$$= \frac{1}{2k^2} \max \left\{ \begin{array}{l} d(Tv,Tv), d(Tv,Tv), d(Tv,Gv), \\ d(Tv,Gv), d(Tv,Tv) \end{array} \right\}$$
$$\leq \max\{d(Tv,Gv), d(Gv,Tv)\}.$$

Thus

$$\int_0^{d(Tv,Gv)} \rho(t)dt \leq \int_0^{\max\{d(Tv,Gv),d(Gv,Tv)\}} \rho(t)dt$$

$$\int_0^{d(Gv,Tv)} \rho(t)dt = \int_0^{d(Gv,Fz)} \rho(t)dt$$
$$\leq h \int_0^{M_2(v,z)} \rho(t)dt$$

$$M_2(v,z) = \frac{1}{2k^2} \max \left\{ \begin{array}{l} d(Tv,Sz), d(Tv,Gv), d(Sz,Fz), \\ d(Tv,Fz), d(Sz,Gv) \end{array} \right\}$$
$$= \frac{1}{2k^2} \max \left\{ \begin{array}{l} d(Tv,Tv), d(Tv,Gv), d(Tv,Tv), \\ d(Tv,Tv), d(Tv,Gv) \end{array} \right\}$$
$$\leq \max\{d(Tv,Gv), d(Gv,Tv)\}.$$

Thus

$$\int_0^{d(Gv,Tv)} \rho(t)dt \leq h \int_0^{\max\{d(Tv,Gv),d(Gv,Tv)\}} \rho(t)dt.$$

Hence we have

$$\int_0^{\max\{d(Tv,Gv),d(Gv,Tv)\}} \rho(t)dt \leq h \int_0^{\max\{d(Tv,Gv),d(Gv,Tv)\}} \rho(t)dt$$

which in turn yields that $Gv = Tv = z$.

Since the pair (G, T) is weakly compatible, we have $Tz = TGv = GTv = Gz$.

From (7.6.1), we have

$$\begin{aligned}\int_0^{d(z,Tz)} \rho(t)dt &= \int_0^{d(Fz,Tz)} \rho(t)dt \\ &\leq h \int_0^{M_1(z,z)} \rho(t)dt\end{aligned}$$

$$\begin{aligned}M_1(z,z) &= \frac{1}{2k^2} \max \left\{ \begin{array}{l} d(Sz,Tz), d(Sz,Fz), d(Tz,Gz), \\ d(Sz,Gz), d(Tz,Fz) \end{array} \right\} \\ &= \frac{1}{2k^2} \max \left\{ \begin{array}{l} d(z,Tz), d(z,z), d(Tz,Tz), \\ d(z,Tz), d(Tz,z) \end{array} \right\} \\ &\leq \max\left\{d(z,Tz), d(Tz,z)\right\}.\end{aligned}$$

Thus

$$\int_0^{d(z,Tz)} \rho(t)dt \quad \leq \quad h \int_0^{\max\{d(z,Tz),d(Tz,z)\}} \rho(t)dt.$$

Similarly we can show that

$$\int_0^{d(Tz,z)} \rho(t)dt \quad \leq \quad h \int_0^{\max\{d(z,Tz),d(Tz,z)\}} \rho(t)dt.$$

Hence we have

$$\int_0^{\max\{d(z,Tz),d(Tz,z)\}} \rho(t)dt \quad \leq \quad h \int_0^{\max\{d(z,Tz),d(Tz,z)\}} \rho(t)dt$$

which in turn yields that $Tz = z$.

Thus $Gz = Tz = z$.

Thus z is a common fixed point of F, G, S and T.

Uniqueness of common fixed point follows easily from (7.6.1) and (7.6.2).

Now we give an example to illustrate Theorem 7.6

Example 7.7. Let $X = [0,1]$ and $d(x,y) = |2x-y|^2 + |2x+y|^2$.
Let $F,G,S,T : x \to X$ be defined by $Fx = \frac{x^2}{128}, Gx = \frac{x^2}{256}, Sx = \frac{x^2}{8}$ and $Tx = \frac{x^2}{16}$.Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\rho(t) = 1$.

As in Example 7.5, d is a dislocated quasi b-metric with $k = 2$.

Consider

$$\begin{aligned}
\int_0^{d(Fx,Gy)} \rho(t)dt &= \int_0^{d(Fx,Gy)} 1dt \\
&= d(Fx,Gy) \\
&= \left|\frac{x^2}{64} - \frac{y^2}{256}\right|^2 + \left|\frac{x^2}{64} + \frac{y^2}{256}\right|^2 \\
&= \frac{1}{16^2}\left[\left|\frac{x^2}{4} - \frac{y^2}{16}\right|^2 + \left|\frac{x^2}{4} + \frac{y^2}{16}\right|^2\right] \\
&= \frac{1}{256}\left[|2Sx - Ty|^2 + |2Sx + Ty|^2\right] \\
&= \frac{1}{256}\left[d(Sx,Ty)\right] \\
&\leq h\frac{1}{2k^2}\max\left\{\begin{array}{l} d(Sx,Ty), d(Sx,Fx), d(Ty,Gy), \\ d(Sx,Gy), d(Ty,Fx) \end{array}\right\} \\
&= hM_1(x,y) \\
&= h\int_0^{M_1(x,y)} 1dt \\
&= h\int_0^{M_1(x,y)} \rho(t)dt.
\end{aligned}$$

Thus (7.6.1) is satisfied. Similarly we can verify (7.6.2).

One can easily verify all the remaining conditions of Theorem 7.6 and $x = 0$ is the unique common fixed point of F, G, S and T.

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Common Fixed Point Theorems for Expansive Mappings in G - Metric Spaces

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ABSTRACT

In this paper we obtain a common fixed point theorem for three expansive mappings and a unique common fixed point theorem for two Jungck type expansive mappings in G-metric spaces.

Key words : Expansive mappings, G- metric space, weakly compatible mappings.
Subject classification: 47 H 10, 54 H 25

1. INTRODUCTION

Dhage[2,3,4,5] *et al.* introduced the concept of D –metric spaces as generalization of ordinary metric functions and went on to present several fixed point results for single and multivalued mappings. Mustafa and Sims[6] and Naidu *et al.*[10,11,12] demonstrated that most of the claims concerning the fundamental topological structure of D – metric space are incorrect . Alternatively, Mustafa and Sims[6] introduced more appropriate notion of generalized metric space or a G – metric space and obtained some topological properties in it . Later Zead Mustafa, Hamed Obiedat and Fadi Awawdeh[7], Mustafa, Shatanawi and Bataineh[8], Mustafa and Sims[9], Shatanawi[13] and Renu Chugh, Tamanna Kadian, Anju Rani and B. E. Rhoades[1] *et al.* obtained some fixed point theorems for a single map in G- metric spaces. In this paper, we obtain a unique common fixed point theorem for six weakly compatible mappings in G – metric spaces. First , we present some known definitions and propositions in G – metric spaces .

Definition 1.1[6]: Let X be a nonempty set and let $G: X \times X \times X \to R^+$ be a function satisfying the following properties :

(G_1) : $G(x, y, z) = 0$, if $x = y = z$,

(G_2) : $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G_3) : $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G_4) : $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, symmetry in all three variables,

(G_5) : $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a generalized metric or a G – metric on X and the pair (X, G) is called a G- metric space.

Definition1. 2 [6] : Let (X, G) be a G- metric space and $\{x_n\}$ be a sequence in X. A point $x \in X$ is said to be limit of $\{x_n\}$ iff $\lim_{n, m \to \infty} G(x, x_n, x_m) = 0$. In this case , the sequence $\{x_n\}$ is said to be G – convergent to x.

Definition 1.3 [6] : Let (X, G) be a G- metric space and $\{x_n\}$ be a sequence in X. $\{x_n\}$ is called G- Cauchy iff $\lim_{l,n,m\to\infty} G(x_l, x_n, x_m) = 0$. (X, G) is called G –complete if every G–Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 1.4 [6] : In a G- metric space,(X, G), the following are equivalent.

(1) The sequence $\{x_n\}$ is G- Cauchy.

(2) For every $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Proposition 1.5 [6] : Let (X, G) be a G- metric space. Then the function G (x, y, z) is jointly continuous in all three of its variables.

Proposition 1.6 [6] : Let (X, G) be a G- metric space. Then for any $x, y, z, a \in X$, it follows that

(i) if $G(x, y, z) = 0$ then $x = y = z$,

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \leq 2G(x, x, y)$,

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \leq \frac{2}{3}[G(x, a, a) + G(y, a, a) + G(z, a, a)]$.

Proposition 1.7 [6] : Let (X, G) be a G- metric space. Then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$, the following are equivalent .

(i) $\{x_n\}$ is G- convergent to x,

(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

2. MAIN RESULTS

Theorem 2.1: Let (X, G) be a complete G-metric space . If there exist a constant $q > 1$ and surjective mappings A, B and C on X such that $G(Ax,By,Cz) \geq q \max \{G(x, y, z), G(x, Ax, Cz), G(y, By, Ax), G(z, Cz, By)\}$ for all $x,y,z \in X$, then

a) A or B or C has a fixed point in X,

(or)

(b) A, B and C has a unique common fixed point in X.

Proof: Let $x_0 \in X$,

There exist $x_1, x_2, x_3 \in X$ such that $x_0 = A x_1$, $x_1 = B x_2$, $x_2 = C x_3$.

By induction, we have

$x_{3n} = A x_{3n+1}$, $x_{3n+1} = B x_{3n+2}$, $x_{3n+2} = C x_{3n+3}$, $n = 0,1,2,\ldots$

If $x_{3n+1} = x_{3n}$ then $Ax = x$, where $x = x_{3n}$.

If $x_{3n+2} = x_{3n+1}$ then $B x = x$, where $x = x_{3n+1}$.

If $x_{3n+3} = x_{3n+2}$ then $C x = x$, where $x = x_{3n+2}$.

Assume that $x_n \neq x_{n+1}$ for all n .

Denote $d_n = G(x_n, x_{n+1}, x_{n+2})$.

$d_{3n-1} = G(x_{3n-1}, x_{3n}, x_{3n+1}) = G(C x_{3n}, A x_{3n+1}, B x_{3n+2})$

$$\geq q \max \left\{ \begin{array}{l} G(x_{3n+1}, x_{3n+2}, x_{3n}), G(x_{3n+1}, x_{3n}, x_{3n-1}), \\ G(x_{3n+2}, x_{3n+1}, x_{3n}), G(x_{3n}, x_{3n-1}, x_{3n+1}) \end{array} \right\}$$

$= q \max \{d_{3n}, d_{3n-1}, d_{3n}, d_{3n-1}\}$.

Thus we have $d_{3n-1} \geq q\, d_{3n}$ so that $d_{3n} \leq k\, d_{3n-1}$ where $k = \frac{1}{q} < 1$. …(1)

$d_{3n} = G(x_{3n}, x_{3n+1}, x_{3n+2}) = G(Ax_{3n+1}, Bx_{3n+2}, Cx_{3n+3})$

$$\geq q \max \left\{ \begin{array}{l} G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n+1}, x_{3n}, x_{3n+2}), \\ G(x_{3n+2}, x_{3n+1}, x_{3n}), G(x_{3n+3}, x_{3n+2}, x_{3n+1}) \end{array} \right\}$$

$= q \max \{d_{3n+1}, d_{3n}, d_{3n}, d_{3n+1}\}$.

Thus we have $d_{3n} \geq q\, d_{3n+1}$ so that $d_{3n+1} \leq k\, d_{3n}$ …(2)

$d_{3n+1} = G(x_{3n+1}, x_{3n+2}, x_{3n+3}) = G(Bx_{3n+2}, Cx_{3n+3}, Ax_{3n+4})$

$\geq q \max$

$$\begin{Bmatrix} G(x_{3n+4}, x_{3n+2}, x_{3n+3}), G(x_{3n+4}, x_{3n+3}, x_{3n+2}), \\ G(x_{3n+2}, x_{3n+1}, x_{3n+3}), G(x_{3n+3}, x_{3n+2}, x_{3n+1}) \end{Bmatrix}$$

$= q \max \{d_{3n+2}, d_{3n+2}, d_{3n+1}, d_{3n+1}\}$

Thus we have $d_{3n+1} \geq q\, d_{3n+2}$ so that $d_{3n+2} \leq k\, d_{3n+1}$ …..(3)

From (1), (2), (3) we have $d_n \leq k\, d_{n-1}$, n = 1,2,3…….

From (G_3), we have

$$\begin{aligned} G(x_n, x_n, x_{n+1}) &\leq G(x_n, x_{n+1}, x_{n+2}) \\ &\leq k\, G(x_{n-1}, x_n, x_{n+1}) \\ &\leq k^2\, G(x_{n-2}, x_{n-1}, x_n) \\ &\;\vdots \\ &\leq k^n\, G(x_0, x_1, x_2). \end{aligned}$$

Now, using (G_5), for m > n

$G(x_n, x_n, x_m) \leq G(x_n, x_n, x_{n+1}) + G(x_{n+1}, x_{n+1}, x_{n+2}) + G(x_{n+2}, x_{n+2}, x_{n+3}) + \ldots + G(x_{m-1}, x_{m-1}, x_m)$

$\leq (k^n + k^{n+1} + k^{n+2} + \ldots + k^{m-1})\, G(x_0, x_1, x_2)$

$$\leq \frac{k^n}{1-k} G(x_0, x_1, x_2)$$

$$\to 0 \text{ as } n \to \infty, m \to \infty.$$

Hence $\{x_n\}$ is G – Cauchy. Since (X, G) is complete, there exists $p \in X$ such that $\{x_n\}$ is G- convergent to p. Now

$G(Ap, x_{3n+1}, x_{3n+2}) = G(Ap, Bx_{3n+2}, Cx_{3n+3})$

$\geq q \max$

$$\begin{Bmatrix} G(p, x_{3n+2}, x_{3n+3}), G(p, Ap, x_{3n+2}), \\ G(x_{3n+2}, x_{3n+1}, Ap), G(x_{3n+3}, x_{3n+2}, x_{3n+1}) \end{Bmatrix}$$

Letting $n \to \infty$, we get

$G(Ap, p, p) \geq q \max \{0, G(p, Ap, p), G(p, p, Ap), 0\}$.

Thus $G(Ap, p, p) = 0$ so that $Ap = p$.

$G(x_{3n}, Bp, x_{3n+2}) = G(Ax_{3n+1}, Bp, Cx_{3n+3})$

$\geq q \max$

$$\begin{Bmatrix} G(x_{3n+1}, p, x_{3n+3}), G(x_{3n+1}, x_{3n}, x_{3n+2}), \\ G(p, Bp, x_{3n}), G(x_{3n+3}, x_{3n+2}, Bp) \end{Bmatrix}$$

Letting $n \to \infty$, we get

$G(p, Bp, p) \geq q \max \{0, 0, G(p, Bp, p), G(p, Bp, p)\}$.

Thus $G(p, Bp, p) = 0$ so that $Bp = p$.

$G(x_{3n}, x_{3n+1}, Cp) = G(Ax_{3n+1}, Bx_{3n+2}, Cp)$

$\geq q \max$

$$\begin{Bmatrix} G(x_{3n+1}, x_{3n+2}, p), G(x_{3n+1}, x_{3n}, Cp), \\ G(x_{3n+2}, x_{3n+1}, x_{3n}), G(p, Cp, x_{3n+1}) \end{Bmatrix}$$

Letting $n \to \infty$, we get

$G(p, p, Cp) \geq q \max \{0, G(p, p, Cp), 0, G(p, Cp, p)\}$.

Thus $G(p, p, Cp) = 0$ so that $Cp = p$.

Thus p is a common fixed point of A, B and C.

Suppose p' is another common fixed point of A, Band C .

$$\begin{aligned} G(p, p, p') &= G(Ap, Bp, Cp') \\ &\geq q \max\{G(p, p, p'), G(p, p, p')0, G(p', p', p)\} \\ &\geq q \max \{G(p, p, p'), \tfrac{1}{2} G(p, p, p')\} \text{ since } G(p, p, p') \leq 2\, G(p', p', p) \\ &= q\, G(p, p, p'). \end{aligned}$$

Hence $p' = p$.

Thus p is a unique common fixed point of A, B and C.

Corollary 2.2: Let (X, G) be a complete G-metric space. If there exist a constant q > 1 and surjective mapping T on X such that $G(Tx, Ty, Tz) \geq q \max\{G(x, y, z), G(x, Tx, Tz), G(y, Ty, Tx), G(z, Tz, Ty)\}$ for all $x, y, z \in X$,

then T has a unique fixed point in X.

Proof: Let $x_0 \in X$.

There exists a sequence $\{x_n\}$ in X such that $x_n = T x_{n+1}$, n = 0, 1, 2…

If $x_n = x_{n+1}$ for some n then T x = x, where $x = x_{n+1}$.

Assume that $x_n \neq x_{n+1}$ for all n.

The rest of the proof follows as in Theorem 2.1.

Theorem 2.3: Let (X, G) be a G – metric space and A, f : X $\longrightarrow$ X be satisfying

(2.3.1) $G(Ax, Ay, Az) \geq q \max \begin{Bmatrix} G(fx, fy, fz), G(fx, Ax, fz), \\ G(fy, Ay, fx), G(fz, Az, fy) \end{Bmatrix}$,

for all x, y, z $\in$ X, where q > 1,

(2.3.2) f (X) $\subseteq$ A(X) and f (X) is a G – complete sub space of X and

(2.3.3) the pair (A, f) is weakly compatible.

Then A and f have a unique common fixed point .

Proof : Let $x_0 \in X$.

From (2.3.2), there exists $x_1 \in X$ such that $fx_0 = Ax_1 = y_1$, say.

Inductively, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$fx_{n-1} = Ax_n = y_n$, n = 1, 2, 3, ….

Case (i): Suppose $y_n = y_{n-1}$ for some n.Then $fx_{n-1} = Ax_{n-1}$.

Thus fp = Ap where $p = x_{n-1}$.

Since (A, p) is weakly compatible, we have $f^2p = f(fp) = f(Ap) = Afp = A^2p$.

$G(A^2p,Ap,Ap) \geq q \max\{G(fAp, fp, fp), G(fAp, AAp, fp), G(fp, Ap, fAp), G(fp, Ap, fp)\}$

$= q \max \{G(A^2p,Ap,Ap),\ G(A^2p,A^2p,Ap),\ G(Ap,Ap,A^2p),\ 0\}$

$\geq q\, G(A^2p, Ap, Ap)$, since $G(A^2p,Ap,Ap) \leq 2\, G(Ap, A^2p, A^2p)$

Hence $A^2p = Ap$. Then $fAp = A^2p = Ap$.

Ap is a common fixed point of f and A.

Case (ii) : Assumethat $y_n \neq y_{n+1}$ for all n

$G(y_{n-1}, y_{n-1}, y_n) = G(Ax_{n-1}, Ax_{n-1}, Ax_n)$

$\geq q \max \begin{Bmatrix} G(y_n, y_n, y_{n+1}), G(y_n, y_{n-1}, y_{n+1}), \\ G(y_n, y_{n-1}, y_n), G(y_{n+1}, y_n, y_n) \end{Bmatrix}$

$\geq q \max$

$$\begin{Bmatrix} G(y_n, y_n, y_{n+1}), G(y_{n-1}, y_{n-1}, y_n), \\ \frac{1}{2} G(y_{n-1}, y_{n-1}, y_n), G(y_n, y_n, y_{n+1}) \end{Bmatrix},$$

since $G(y_{n-1}, y_{n-1}, y_n) \leq G(y_{n-1}, y_n, y_{n+1})$

and $G(y_{n-1}, y_{n-1}, y_n) \leq 2\, G(y_{n-1}, y_n, y_n)$.

Thus $G(y_{n-1}, y_{n-1}, y_n) \geq q\, G(y_n, y_n, y_{n+1})$.

Hence $G(y_n, y_n, y_{n+1}) \leq k\, G(y_{n-1}, y_{n-1}, y_n)$,

where $k = \frac{1}{q} < 1$

$\leq k^2\, G(y_{n-2}, y_{n-2}, y_{n-1})$

$\leq k^3\, G(y_{n-3}, y_{n-3}, y_{n-2})$

:

:

$\leq k^n\, G(y_0, y_0, y_1)$.

Now, using (G_5), for m < n we have

$G(y_n, y_n, y_m) \leq G(y_n, y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_{n+2}) + \ldots + G(y_{m-1}, y_{m-1}, y_m)$

$\leq (k^n + k^{n+1} + \ldots + k^{m-1})\, G(y_0, y_0, y_1)$

$\leq \frac{k^n}{1-k}\, G(y_0, y_0, y_1)$

$\to 0$ as $n \to \infty$, $m \to \infty$.

Hence $\{y_n\}$ is G-Cauchy.

Since f(X) is G- complete, there exists p, t $\in$ X such that $y_n \to p = ft$.

$G(At, y_n, y_m) = G(At, Ax_n, Ax_n)$

$\geq q \max\{G(p, y_{n+1}, y_{n+1}), G(p, At, y_{n+1}), G(y_{n+1}, y_n, p), G(y_{n+1}, y_n, y_n)\}$

Letting $n \to \infty$, we get

$G(At, p, p) \geq q\, G(p, At, p)$

Thus A t = p. Hence f t = A t.

As in case (i), f t (= A t = p) is the unique common fixed point of f and A .

Uniqueness: Suppose p' is another common fixed point of A and f.

$G(p, p, p') = G(Ap, Ap, Ap')$

$\geq q \max \{G(p, p, p'), G(p, p, p'),$

$0, G(p', p', p)\}$

$\geq q \max \{G(p, p, p'), \frac{1}{2} G(p, p, p')\}$

$= q\, G(p, p, p').$

Hence $p' = p$.

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COINCIDENCE POINT THEOREM FOR TWO PAIRS OF HYBRID MAPPINGS IN COMPLEX VALUED METRIC SPACES

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Abstract. In this paper using f is S-Weakly commuting we prove a coincidence point theorem for two pairs of hybrid mappings in a complex valued metric space.Our theorem is a generalization of Theorem 10 of Azam,Ahmad and Kumam [2].

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1. Introduction

It is a well-known fact that the mathematical results regarding fixed points of contraction type mappings are very useful for determining the existence and uniqueness of solutions to various mathematical models.Over the last 40 years,the theory of fixed points has been developed regarding the results that are related to finding the fixed points of self and nonself nonlinear mappings in a metric space.

The study of fixed points for multi-valued contraction mappings was initiated by Nadler[18] and Markin[8].Several authors proved fixed point results in different types of generalized metric spaces[1, 3, 5, 7, 10, 11, 12, 13, 14, 15, 16, 17, 19].

Azam et al.[1] introduced the concept of a complex valued metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive type condition.Subsequently,Rouzkard and Imdad [6] established some common fixed point theorems for maps satisfying certain rational expressions in complex valued metric spaces to generalize the results of [1].In the same way, Sintunavarat et al. [21, 22] obtained common fixed point results by replacing the constant of contractive condition to control functions.Recently ,Sitthikul and Saejung [9] and Klin-eam and Suanoom [4] established some fixed point results by generalizing the contractive conditions in the context of complex valued metric spaces.Very recently,Ahmad et al.[7] obtained some new fixed point results for multi-valued mappings in the setting of complex valued metric spaces.

Throughout this paper,N and C denote the set of all positive integers and the set of all complex numbers respectively.

A complex number $z \in C$ is an ordered pair of real numbers, whose first co-ordinate is called $Re(z)$ and second co-ordinate is called $Im(z)$. Let $z_1, z_2 \in C$. Define apartial order $\precsim$ on C as follows:
$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$.
Thus $z_1 \precsim z_2$ if one of the following holds:
(1)$Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(2)$Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(3)$Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(4)$Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
We will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2), (3) and (4) is satisfed;also we will write $z_1 \prec z_2$ if only (4) is satisfed.

Definition 1.1. ([1]) Let X be a non empty set. A function $d : X \times X \to \mathbf{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:
(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \precsim d(x, z) + d(z, y)$.
The pair $(X, d)$is called a complex valued metric space.

Let $\{x_n\}$ be a sequence in X and $x \in X$.If for every $c \in C$ with $0 \preceq c$ there is $n_0 \in N$ such that for all $n > n_0, d(x_n, x) \prec c$,then $\{x_n\}$ is said to be convergent to x and x is called the limit point of $\{x_n\}$.We denote this by $lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.If for every $c \in C$ with $0 \prec c$ there is $n_0 \in N$ such that for all $n > n_0, d(x_n, x_{n+m}) \prec c$, where $m \in N$,then $\{x_n\}$ is called Cauchy sequence in(X, d).If every Cauchy sequence is convergent in (X, d) then (X, d) is called a complete complexvalued metric space.
We require the follwing lemmas.
The following lemmas are very useful for further discussion.

Lemma 1.2. *([1]) Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.3. *([1]) Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n, m \to \infty$.*

Now we follow the notations and definitions given in [7].

Let (X,d) be a complex valued metric space. We denote
$s(z_1) = \{z_2 \in C : z_1 \precsim z_2\}$ for $z_1 \in C$ and
$s(a, B) = \bigcup_{b \in B} s(d(a,b)) = \bigcup_{b \in B} \{z \in C : d(a,b) \precsim z\}$ for $a \in X$ and $B \in C(X)$.
For $A, B \in C(X)$, we denote
$s(A,B) = \left(\bigcap_{a \in A} s(a,B)\right) \cap \left(\bigcap_{b \in B} s(b,A)\right)$.

Remark 1.4. ([7]) Let (X,d) be a complex valued metric space and let CB(X) be a collection of nonempty closed subsets of X. Let $T : X \to CB(X)$ be a multivaluedmap.For $x \in X$ and $A \in CB(X)$,
define $W_x(A) = \{d(x,a) : a \in A\}$.
Thus , for $x, y \in X$. $W_x(Ty) = \{d(x,u) : u \in Ty\}$.

Definition 1.5. ([7]) Let (X,d) be a complex valued metric space.A nonempty subset A of X is called bounded from below if there exists some $z \in C$ such that $z \precsim a$ for all $a \in A$.

Definition 1.6. ([7]) Let (X,d) be a complex valued metric space. A multivalued mapping $F : X \to 2^C$ is called bounded from below if for each $x \in X$ there exists $z_x \in C$ such that $z_x \precsim u$ for all $u \in Fx$.

Definition 1.7. ([7]) Let (X,d) be a complex valued metric space.The multivalued mapping $T : X \to CB(X)$ is said to have the lower bound proerty (l.b.Property) on (X,d) if the for any $x \in X$, the multi-valued mapping $F_x : X \to 2^C$ defined by $F_x(y) = W_x(Ty)$ is bounded from below.That is for $x, y \in X$,there exists an element $l_x(Ty) \in C$ such that $l_x(Ty) \precsim u$, for all $u \in W_x(Ty)$, where $l_x(Ty)$ is called a lower bound of T associated with (x,y).

Definition 1.8. ([7]) Let (X,d) be a complex valued metric space.The multivalued mapping $T : X \to CB(X)$ is said to have the gretest lower bound proerty (g.l.b.Property) on (X,d) if the gretest lower bound of $W_x(Ty)$ exists in C for all $x, y \in X$. We denote $d(x,Ty)$ by the g.l.b.Property of $W_x(Ty)$. That is $d(x,Ty) = inf\{d(x,u) : u \in Ty\}$.

Definition 1.9. ([20]) Let $f : X \to X, S : X \to CB(X)$.$f$ is said to be S-weakly commuting at $x \in X$ if $f^2x \in Sfx$.

2. Main Results

Theorem 2.1. *Let (X, d) be a complex valued metric space.*
Let $S, T : X \to CB(X)$ be multi valued mappings $f, g : X \to X$ satisfying

(2.1.1)$Sx \subseteq g(X)$,$Tx \subseteq f(X)$,$\forall x \in X$

(2.1.2)$ad(fx, Ty) + bd(gy, Sx) + \frac{cd(fx,Ty)d(gy,Sx)}{1+d(fx,gy)} \in s(Sx, Ty)$

for all $x, y \in X$ and a, b, c are non negative reals such that $2a + 2b < 1$,

(2.1.3)f *is* S *weakly commuting and* g *is* T *weakly commuting,*

(2.1.4)$f(X)$ *is complete.*

Then (f,S) and (g,T) have the same coincidence point.

Proof. Let x_1 be an arbitrary point in X. Write $y_1 = fx_1$.Since $Sx_1 \subseteq g(X)$, there exists $x_2 \in X$ such that $y_2 = gx_2 \in Sx_1$.
From (2.1.2) ,we have

$ad(fx_1, Tx_2) + bd(gx_2, Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)} \in s(Sx_1, Tx_2)$.

$ad(fx_1, Tx_2) + bd(gx_2, Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)} \in \left(\bigcap_{x \in Sx_1} s(x, Tx_2) \right)$.

$ad(fx_1, Tx_2) + bd(gx_2, Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)} \in s(x, Tx_2), \forall x \in Sx_1$.

$ad(fx_1, Tx_2) + bd(gx_2, Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)} \in s(gx_2, Tx_2)$.

$ad(fx_1, Tx_2) + bd(gx_2, Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)} \in \bigcup_{x \in Tx_2} s\left(d(gx_2, x)\right)$.

Since $Tx_2 \subseteq f(X)$,there exists some $x_3 \in X$ with $y_3 = fx_3 \in Tx_2$ suchthat $ad(fx_1, Tx_2) + bd(gx_2, Sx_1) +$
$\in s\left(d(gx_2, fx_3)\right)$.

Hence

$d(gx_2, fx_3) \precsim ad(fx_1, Tx_2) + bd(gx_2, Sx_1) + \frac{cd(fx_1,Tx_2)d(gx_2,Sx_1)}{1+d(fx_1,gx_2)}$.

$d(y_2, y_3) \precsim ad(y_1, y_3) + bd(y_2, y_2) + \frac{cd(y_1,y_3)d(y_2,y_2)}{1+d(y_1,y_2)}$.

$|d(y_2, y_3)| \leq a\,|d(y_1, y_2)| + a\,|d(y_2, y_3)|$.

$|d(y_2, y_3)| \leq \frac{a}{1-a}\,|d(y_1, y_2)|$. '.....(1)

Now,

$ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)} \in s(Sx_3, Tx_2)$.

$ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)}. \in \left(\bigcap_{y \in Tx_2} s(Sx_3, y) \right)$.

$ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)}. \in s(Sx_3, y), \forall y \in Tx_2$

$ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)} \in s(Sx_3, fx_3).$

$ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)} \in \bigcup_{y \in Sx_3} s(d(y, fx_3)).$

Since $Sx_3 \subseteq g(X)$,there exists some $x_4 \in X$ with $y_4 = gx_4 \in Sx_3$ such that $ad(fx_3, Tx_2) + bd(gx_2, Sx_3)+$ $\in s(d(gx_4, fx_3)).$

Hence

$d(gx_4, fx_3) \precsim ad(fx_3, Tx_2) + bd(gx_2, Sx_3) + \frac{cd(fx_3,Tx_2)d(gx_2,Sx_3)}{1+d(fx_3,gx_2)}.$

$d(y_3, y_4) \precsim ad(y_3, y_3) + bd(y_2, y_4) + \frac{cd(y_3,y_3)d(y_2,y_4)}{1+d(y_3,y_2)}.$

$|d(y_3, y_4)| \leq b\,|d(y_2, y_3)| + b\,|d(y_3, y_4)|$

$\left|d(y_{3,}y_4)\right| \leq \frac{b}{1-b}\left|d(y_{2,}y_3)\right|.$ ......(2)

putting $h = max\left\{\frac{a}{1-a}, \frac{b}{1-b}\right\}$ and we continuing in this way, we get

$$\begin{aligned}
\left|d(y_n, y_{n+1})\right| &\leq h\left|d(y_{n-1}, y_n)\right| \\
&\leq h^2\left|d(y_{n-2}, y_{n-1})\right| \\
&. \\
&. \\
&. \\
&\leq h^{n-1}\left|d(y_1, y_2)\right|
\end{aligned}$$

Now for $m > n$ consider

$$\begin{aligned}
\left|d(y_{n,}y_m)\right| &\leq \left|d(y_{n,}y_{n+1}) + d(y_{n+1,}y_{n+2}) + ... + d(y_{m-1,}y_m)\right| \\
&\leq h^{n-1} + h^n + ... + h^{m-2}\left|d(y_{1,}y_2)\right| \\
&\leq \left[\frac{h^{n-1}}{1-h}\right] \to\ 0\ as\ m, n \to \infty.
\end{aligned}$$

Thus $\{y_n\}$ is a Cauchy sequence in X.

Since $f(X)$ is complete , $\{y_{2n+1}\} = \{fx_{2n+1}\}$ is Cauchy,it follows that $\{y_{2n+1}\}$ converges to $u \in f(X)$.Hence there exists $v \in X$ such that $u = fv$.

Since $\{y_n\}$ is a Cauchy sequence and $\{y_{2n+1}\} \to u$ it follow that $\{y_{2n}\} \to u$.

$ad(fv, Tx_{2n}) + bd(gx_{2n}, Sv) + \frac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})} \in s(Sv, Tx_{2n}).$

$ad(fv, Tx_{2n}) + bd(gx_{2n}, Sv) + \frac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})} \in \left(\bigcap_{y \in Tx_{2n}} s(Sv, y)\right).$

$ad(fv, Tx_{2n}) + bd(gx_{2n}, Sv) + \frac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})} \in s(Sv, y), \forall y \in Tx_{2n}.$

$ad(fv, Tx_{2n}) + bd(gx_{2n}, Sv) + \frac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})} \in s(Sv, y_{2n+1}).$

$ad(fv,Tx_{2n})+bd(gx_{2n},Sv)+\frac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})}\in\bigcup_{u^1\in Sv}s\left(d(u^1,y_{2n+1})\right).$

There exists $v_n\in Sv$ such that

$ad(fv,Tx_{2n})+bd(gx_{2n},Sv)+\frac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})}\in s\left(d(v_n,y_{2n+1})\right).$

Therefore $d(v_n,y_{2n+1})\precsim ad(fv,Tx_{2n})+bd(gx_{2n},Sv)+\frac{cd(fv,Tx_{2n})d(gx_{2n},Sv)}{1+d(fv,gx_{2n})}.$

Using g.l.b.property, we get

$d(v_n,y_{2n+1})\preceq ad(fv,y_{2n+1})+bd(y_{2n},v_n)+\frac{cd(fv,y_{2n+1})d(y_{2n},v_n)}{1+d(fv,y_{2n})}.$

Using triangular inequality, we obtain

$d(v_n,y_{2n+1})\precsim ad(fv,y_{2n+1})+bd(y_{2n},y_{2n+1})+bd(y_{2n+1},v_n)+\frac{cd(fv,y_{2n+1})d(y_{2n},v_n)}{1+d(fv,y_{2n})}.$

$d(v_n,y_{2n+1})\precsim\frac{a}{1-b}d(fv,y_{2n+1})+\frac{b}{1-b}d(y_{2n},y_{2n+1})+\frac{c}{1-b}\frac{d(fv,y_{2n+1})d(y_{2n},v_n)}{1+d(fv,y_{2n})}.$

Now consider

$d(fv,v_n)\precsim d(fv,y_{2n+1})+d(y_{2n+1},v_n).$

$\precsim d(fv,y_{2n+1})+\frac{a}{1-b}d(fv,y_{2n+1})+\frac{b}{1-b}d(y_{2n},y_{2n+1})+\frac{c}{1-b}\frac{d(fv,y_{2n+1})d(y_{2n},v_n)}{1+d(fv,y_{2n})}$

$|d(fv,v_n)|\leq|d(fv,y_{2n+1})|+\frac{a}{1-b}|d(fv,y_{2n+1})|+\frac{b}{1-b}|d(y_{2n},y_{2n+1})|$
$+\frac{c}{1-b}\frac{|d(fv,y_{2n+1})|\,|d(y_{2n},v_n)|}{|1+d(fv,y_{2n})|}.$ Letting $n\to\infty$,

we obtain

$|d(fv,v_n)|\to 0$ as $n\to\infty$.By Lemma 1.2,we have $v_n\to fv$ as $n\to\infty$.

Since Sv is closed and $\{v_n\}\subseteq Sv$, it follows that $fv\in Sv$.

Now $u=fv\in Sv$ and $Sv\subseteq g(X)$ it follows that $u=fv=gw$ for some $w\in X$.

$ad(fx_{2n-1},Tw)+bd(gw,Sx_{2n-1})+\frac{cd(fx_{2n-1},Tw)d(gw,Sx_{2n-1})}{1+d(fx_{2n-1},gw)}\in s(Sx_{2n-1},Tw).$

$ad(fx_{2n-1},Tw)+bd(gw,Sx_{2n-1})+\frac{cd(fx_{2n-1},Tw)d(gw,Sx_{2n-1})}{1+d(fx_{2n-1},gw)}.$
$\in\left(\bigcap_{y^1\in Sx_{2n-1}}s(y^1,Tw)\right).$

$ad(fx_{2n-1},Tw)+bd(gw,Sx_{2n-1})+\frac{cd(fx_{2n-1},Tw)d(gw,Sx_{2n-1})}{1+d(fx_{2n-1},gw)}.$
$\in s(y^1,Tw),\forall y^1\in Sx_{2n-1}.$

$ad(fx_{2n-1},Tw)+bd(gw,Sx_{2n-1})+\frac{cd(fx_{2n-1},Tw)d(gw,Sx_{2n-1})}{1+d(fx_{2n-1},gw)}\in s(y_{2n},Tw).$

$ad(fx_{2n-1},Tw)+bd(gw,Sx_{2n-1})+\frac{cd(fx_{2n-1},Tw)d(gw,Sx_{2n-1})}{1+d(fx_{2n-1},gw)}\in\bigcup_{u^1\in Tw}s\left(d(y_{2n},u^1)\right).$

There exists some $w_n\in Tw$ such that

$ad(fx_{2n-1},Tw)+bd(gw,Sx_{2n-1})+\frac{cd(fx_{2n-1},Tw)d(gw,Sx_{2n-1})}{1+d(fx_{2n-1},gw)}\in s(d(y_{2n},w_n)).$

$d(y_{2n}, w_n) \precsim ad(fx_{2n-1}, Tw) + bd(gw, Sx_{2n-1}) + \frac{cd(fx_{2n-1},Tw)d(gw,Sx_{2n-1})}{1+d(fx_{2n-1},gw)}$.

Using g.l.b.property, we obtain

$d(y_{2n}, w_n) \precsim ad(y_{2n-1}, w_n) + bd(gw, y_{2n}) + \frac{cd(y_{2n-1},w_n)d(gw,y_{2n})}{1+d(y_{2n-1},gw)}$.

Using triangular inequality, we have

$d(y_{2n}, w_n) \precsim ad(y_{2n-1}, y_{2n}) + ad(y_{2n}, w_n) + bd(gw, y_{2n}) + \frac{cd(y_{2n-1},w_n)d(gw,y_{2n})}{1+d(y_{2n-1},gw)}$.

$$d(y_{2n}, w_n) \precsim \frac{a}{1-a} d(y_{2n-1}, y_{2n}) + \frac{b}{1-a} d(gw, y_{2n}) + \frac{c}{1-a} \frac{d(y_{2n-1},w_n)d(gw,y_{2n})}{1+d(y_{2n-1},gw)} \ .$$

Now consider $d(gw, w_n) \precsim d(gw, y_{2n}) + d(y_{2n}, w_n)$.

$$\precsim d(gw, y_{2n}) + \frac{a}{1-a} d(y_{2n-1}, y_{2n}) + \frac{b}{1-a} d(gw, y_{2n}) + \frac{c}{1-a} \frac{d(y_{2n-1},w_n)d(gw,y_{2n})}{1+d(y_{2n-1},gw)}.$$

$$|d(gw, w_n)| \leq |d(gw, y_{2n})| + \frac{a}{1-a} |d(y_{2n-1}, y_{2n})| + \frac{b}{1-a} |d(gw, y_{2n})| + \frac{c}{1-a} \frac{|d(y_{2n-1},w_n)||d(gw,y_{2n})|}{|1+d(y_{2n-1},gw)|}.$$

Letting $n \to \infty$ we get

$|d(gw, w_n)| \to 0$ as $n \to \infty$.By Lemma 1.2, we have $w_n \to gw$ as $n \to \infty$.

Since Tw is closed and $\{w_n\} \subseteq Tw$, it follows that $gw \in Tw$.

We have $u = fv = gw \in Tw$.

Since f is S-weakly commuting and g is T-weakly commuting we have

$f^2v \in Sfv \Rightarrow fu \in Su$ and $g^2w \in Tgw \Rightarrow gu \in Tu$.

Thus the pairs (f, S) and (g, T) have the same coincident point. □

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A Unique Common Fixed Point Theorem for Four Mappings Satisfying Presic Type Condition in Fuzzy Metric Spaces

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Abstract In this paper, we obtain a Presic type fixed point theorem for two pairs of jointly $2k$-weakly compatible maps in fuzzy metric spaces.We also give an example to illustrate our main theorem. We obtain two corollaries for three maps and two maps.

Keywords: Fuzzy metric spaces, presic type theorem, jointly $2k$-weakly compatible mappings.

1 Introduction and Preliminaries

There are a number of generalizations of Banach contraction principle. One such generalization is given by S.B.Presic [9] in 1965.

Let $f : X^k \to X$, where $k \geq 1$ is a positive integer. A point $x^* \in X$ is called a fixed point of f if $x^* = f(x^*, x^*, \cdots, x^*)$. Consider the $k-$order non linear difference equation

$$x_{n+1} = f(x_{n-k+1}, x_{n-k+2}, \cdots, x_n) \text{ for } n = k-1, k, k+1, \tag{1.1}$$

with the initial values $x_0, x_1, x_2, \cdots, x_{k-1} \in X$.

Equation (1.1) can be studied by means of fixed point theory in view of the fact that $x \in X$ is a solution of (1.1) if and only if x is a fixed point of f. One of the most important results in this direction is obtained by Presic [9] in the following way. Throughout this paper,let N denote the set of all positive integers.

Theorem 1.1. ([9]) *Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$. Suppose that*

$$d(f(x_1, x_2, \cdots, x_k), f(x_2, x_3, \cdots, x_{k+1})) \leq \sum_{i=1}^{k} q_i \; d(x_i, x_{i+1})$$

holds for all $x_1, x_2, \cdots, x_k, x_{k+1} \in X$, where $q_i \geq 0$ and $\sum_{i=1}^{k} q_i \in [0, 1)$. Then f has a unique fixed point x^*. Moreover, for any arbitrary points $x_1, x_2, \cdots, x_{k+1}$ in X, the sequence $\{x_n\}$ defined by $x_{n+k} = f(x_n, x_{n+1}, \cdots, x_{n+k-1})$, for all $n \in N$ converges to x^*.

Later Ciric and Presic [6] generalized the above theorem as follows.

Theorem 1.2. *([6]). Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$. Suppose that*

$$d(f(x_1, x_2, \cdots, x_k), f(x_2, x_3, \cdots, x_{k+1})) \leq \lambda \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}$$

holds for all $x_1, x_2, \cdots, x_k, x_{k+1}$ in X, where $\lambda \in [0, 1)$. Then f has a fixed point $x^* \in X$. Moreover, for any arbitrary points $x_1, x_2, \cdots, x_{k+1}$ in X, the sequence $\{x_n\}$ defined by $x_{n+k} = f(x_n, x_{n+1}, \cdots, x_{n+k-1})$, for all $n \in N$ converges to x^*. Moreover, if $d(f(u, u, \cdots, u), f(v, v, \cdots, v)) < d(u, v)$ holds for all $u, v \in X$ with $u \neq v$, then x^* is the unique fixed point of f.

Recently Rao et al.[4,5] obtained some Presic type theorems for two and three maps in metric spaces. Now we give the following definition of [4,5].

Definition 1.3. *Let X be a non empty set and $T : X^{2k} \to X$ and $f : X \to X$. The pair (f, T) is said to be $2k$-weakly compatible if $f(T(x, x, ..., x, x)) = T(fx, fx, ..., fx, fx)$ whenever $x \in X$ such that $fx = T(x, x, ..., x, x)$.*

Using this definition, Rao et al. [4] proved the following

Theorem 1.4. *([4]). Let (X, d) be a metric space, k a positive integer and $S, T : X^{2k} \to X, f : X \to X$ be mappings satisfying:*

(1.4.1) $d(S(x_1, x_2, \cdots, x_{2k}), T(x_2, x_3, \cdots, x_{2k+1})) \leq \lambda \max\limits_{1\leq i\leq 2k} \{d(fx_i, fx_{i+1})\}$ *for all* $x_1, x_2, \cdots, x_{2k}, x_{2k+1}$ *in* X,
(1.4.2) $d(T(y_1, y_2, \cdots, y_{2k}), S(y_2, y_3, \cdots, y_{2k+1})) \leq \lambda \max\limits_{1\leq i\leq 2k} \{d(fy_i, fy_{i+1})\}$ *for all* $y_1, y_2, \cdots, y_{2k}, y_{2k+1}$ *in* X, *where* $0 < \lambda < 1$
(1.4.3) $d(S(u, \cdots, u), T(v, \cdots, v)) < d(fu, fv)$, *for all* $u, v \in X$ *with* $u \neq v$
(1.4.4) *Suppose that $f(X)$ is complete and either (f, S) or (f, T) is a $2k-$ weakly compatible pair.*

Then there exists a unique point $p \in X$ such that $fp = p = S(p, \cdots, p) = T(p, \cdots, p)$.

Recently Murthy and Rashmi [8] defined the following function

Definition 1.5. *Let $\phi : [0, 1] \longrightarrow [0, 1]$ be such that:*

(i) ϕ is increasing and continuous function in each variable,
(ii) $\phi(t, t, t, ..., t) \geq t$ for all $t \in [0, 1]$.

Using this function,Murthy and Rashmi [8] extended Theorem 1.4 to fuzzy metric spaces as follows.

Theorem 1.6. ([8]) *Let $(X, M, *)$be a fuzzy metric space and $S, T : X^{2k} \to X, f : X \to X$ be mappings satisfying for each positive integer $k, 0 < q < \frac{1}{2}$ and $t \in [0, \infty)$:*

(1.6.1) $M(S(x_1, x_2, ..., x_{2k-1}, x_{2k}), T(x_2, x_3, ..., x_{2k}, x_{2k+1}), qt) \geq \phi(M(fx_1, fx_2, t), \cdots, M(fx_{2k}, fx_{2k+1}, t))$ *for all* $x_1, x_2, ..., x_{2k+1} \in X$,
(1.6.2) $M(T(y_1, y_2, ..., y_{2k-1}, y_{2k}), S(y_2, y_3, ..., y_{2k}, y_{2k+1}), qt) \geq \phi(M(fy_1, fy_2, t), \cdots, M(fy_{2k}, fy_{2k+1}, t))$ *for all* $y_1, y_2, ..., y_{2k+1} \in X$,
(1.6.3) $M(S(u, u, ..., u, u), T(v, v, ..., v, v), qt) > M(fu, fv, t)$ *for all* $u, v \in X$ *with* $u \neq v$.

Suppose that $f(X)$ is complete and either (f, S) or (f, T) is $2k$-weakly compatible pair.
Then there exists a unique $p \in X$ such that $p = fp = S(p, p, ..., p, p) = T(p, p, ..., p, p)$.

In this paper, we obtain a Presic type theorem for four mappings satisfying a slight different contractive condition in fuzzy metric spaces. We also give an example and two corollaries to our main theorem.

First we recall some basic definitions and lemmas which play crucial roles in the theory of fuzzy metric spaces.

Definition 1.7. *([2]). A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions:*

1. *$*$ is associative and commutative,*
2. *$*$ is continuous,*
3. *$a * 1 = a$ for all $a \in [0, 1]$,*
4. *$a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.*

Two typical examples of a continuous t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.8. *([1]). A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,*

(M_1) $M(x, y, t) > 0$,

(M_2) $M(x,y,t) = 1$ *if and only if* $x = y$,
(M_3) $M(x,y,t) = M(y,x,t)$,
(M_4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,
(M_5) $M(x,y,.) : (0,\infty) \longrightarrow [0,1]$ *is continuous.*

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the *open ball* $B(x,r,t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by $B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}$.

If $(X, M, *)$ is a fuzzy metric space, let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable.

A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence in the sense of [7] if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$, for all $t > 0$ and each positive integer p. The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent.

Example 1.9. *Let* $X = [0,1]$ *and* $a*b = ab$ *for all* $a, b \in [0,1]$ *and let* M *be the fuzzy set on* $X \times X \times (0,\infty)$ *defined by* $M(x,y,t) = e^{-\frac{|x-y|}{t}}$ *for all* $t > 0$. *Then* $(X, M, *)$ *is a fuzzy metric space.*

Lemma 1.10. *([7]). Let* $(X, M, *)$ *be a fuzzy metric space. Then* $M(x,y,t)$ *is non-decreasing with respect to* t*, for all* $x, y \in X$.

Definition 1.11. *([3]). Let* $(X, M, *)$ *be a fuzzy metric space. Then* M *is said to be continuous on* $X^2 \times (0,\infty)$ *if* $\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x,y,t)$, *whenever a sequence* $\{(x_n, y_n, t_n)\}$ *in* $X^2 \times (0,\infty)$ *converges to a point* $(x,y,t) \in X^2 \times (0,\infty)$*.i.e.* $\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1$ *and* $\lim_{n\to\infty} M(x, y, t_n) = M(x,y,t)$.

Lemma 1.12. *([3]). Let* $(X, M, *)$ *be a fuzzy metric space. Then* M *is a continuous function on* $X^2 \times (0,\infty)$.

Now we state the condition (A): $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x, y \in X$.

We observed that in the proof of Theorem 1.6, the authors Murthy and Rashmi [8] inherently used the condition(A).

Now we introduce the definition of jointly $2k$-weakly compatible pairs as follows.

Definition 1.13. *Let* X *be a nonempty set,* k *a positive integer and* $S, T : X^{2k} \to X$ *and* $f, g : X \to X$. *The pairs* (f, S) *and* (g, T) *are said to be jointly* $2k$*-weakly compatible if*

$$f(S(x,x,...,x)) = S(fx, fx, ..., fx)$$

and

$$g(T(x,x,...,x)) = T(gx, gx, ..., gx)$$

whenever there exists $x \in X$ *such that* $fx = S(x,x,...,x)$ *and* $gx = T(x,x,...,x)$.

Now we give our main theorem.

2 Main Result

Throughout this section assume ϕ as in Definition 1.5

Theorem 2.1. *Let* $(X, M, *)$ *be a fuzzy metric space with the condition (A),* k *a positive integer and* $S, T : X^{2k} \longrightarrow X$ *and* $f, g : X \longrightarrow X$ *be mappings satisfying:*

(2.1.1) $S(X^{2k}) \subseteq g(X), T(X^{2k}) \subseteq f(X)$,

(2.1.2) $$M(S(x_1, x_2, ..., x_{2k}), T(y_1, y_2, ..., y_{2k}), qt) \geq \phi\begin{pmatrix} M(gx_1, fy_1, t), M(fx_2, gy_2, t), \\ M(gx_3, fy_3, t), M(fx_4, gy_4, t), \\ \vdots \\ M(gx_{2k-1}, fy_{2k-1}, t), M(fx_{2k}, gy_{2k}, t) \end{pmatrix}$$

$\forall x_1, x_2, ..., x_{2k}, y_1, y_2, .., y_{2k} \in X, \forall t > 0, 0 < q < 1,$

(2.1.3) *(f,S) and (g,T) are jointly $2k$-weakly compatible pairs.*
(2.1.4) *Suppose $z = fu = gu$ for some $u \in X$ whenever there exists a sequence $\{y_{2k+n}\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} y_{2k+n} = z \in X$.*

Then z is the unique point in X such that $z = fz = gz = S(z,z,..,z,z) = T(z,z,...,z,z)$.

Proof. Suppose $x_1, x_2, ..., x_{2k}$ are arbitrary points in X. From (2.1.1),we define

$$y_{2k+2n-1} = S(x_{2n-1}, x_{2n}, ..., x_{2k+2n-2}) = gx_{2k+2n-1}$$

$$y_{2k+2n} = T(x_{2n}, x_{2n+1}, ..., x_{2k+2n-1}) = fx_{2k+2n}$$

for $n = 1, 2, ...$
Let $\alpha_{2n} = M(fx_{2n}, gx_{2n+1}, qt)$ and $\alpha_{2n-1} = M(gx_{2n-1}, fx_{2n}, qt)$ for $n = 1, 2, ...$
Put $\theta = \frac{1}{q}$ and $\mu = min\{\theta\frac{1+\sqrt{\alpha_1}}{1-\sqrt{\alpha_1}}, \theta^2\frac{1+\sqrt{\alpha_2}}{1-\sqrt{\alpha_2}}, ..., \theta^{2k}\frac{1+\sqrt{\alpha_{2k}}}{1-\sqrt{\alpha_{2k}}}\}$.Then $\theta > 1$.
By the selection of μ, we have

$$\alpha_n \geq \left(\frac{\mu - \theta^n}{\mu + \theta^n}\right)^2 for \ \ n = 1, 2, ..., 2k \tag{1}$$

Consider

$$\begin{aligned}
\alpha_{2k+1} &= M(gx_{2k+1}, fx_{2k+2}, qt) \\
&= M(S(x_1, x_2, ..., x_{2k-1}, x_{2k}), T(x_2, x_3, ..., x_{2k}, x_{2k+1}), qt) \\
&\geq \phi(M(gx_1, fx_2, t), M(fx_2, gx_3, t), ..., M(fx_{2k}, gx_{2k+1}, t)) \\
&\geq \phi(\alpha_1, \alpha_2, ..., \alpha_{2k-1}, \alpha_{2k}) \ \text{, since } M(x,y,.) \text{ and } \phi \text{ are increasing} \\
&\geq \phi\left(\left(\frac{\mu-\theta}{\mu+\theta}\right)^2, \left(\frac{\mu-\theta^2}{\mu+\theta^2}\right)^2, ..., \left(\frac{\mu-\theta^{2k}}{\mu+\theta^{2k}}\right)^2\right) \ \text{from (1)} \\
&\geq \phi\left(\left(\frac{\mu-\theta^{2k}}{\mu+\theta^{2k}}\right)^2, \left(\frac{\mu-\theta^{2k}}{\mu+\theta^{2k}}\right)^2, ..., \left(\frac{\mu-\theta^{2k}}{\mu+\theta^{2k}}\right)^2\right) \\
&\geq \left(\frac{\mu-\theta^{2k}}{\mu+\theta^{2k}}\right)^2 \ \text{, since } \phi\,(t,t,...,t) \geq t \\
&\geq \left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2.
\end{aligned}$$

Thus

$$\alpha_{2k+1} \geq \left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2 \tag{2}$$

Also

$$\begin{aligned}
\alpha_{2k+2} &= M(fx_{2k+2}, gx_{2k+3}, qt) \\
&= M(S(x_3, x_4, x_5, x_6, ..., x_{2k+1}, x_{2k+2}), T(x_2, x_3, x_4, x_5, ..., x_{2k}, x_{2k+1}), qt) \\
&\geq \phi(M(gx_3, fx_2, t), M(fx_4, gx_3, t), ...,, M(fx_{2k+2}, gx_{2k+1}, t)) \\
&\geq \phi(\alpha_2, \alpha_3, \alpha_4, \alpha_5, ..., \alpha_{2k}, \alpha_{2k+1}) \\
&\geq \phi\left(\left(\frac{\mu-\theta^2}{\mu+\theta^2}\right)^2, \left(\frac{\mu-\theta^3}{\mu+\theta^3}\right)^2, ..., \left(\frac{\mu-\theta^{2k}}{\mu+\theta^{2k}}\right)^2, \left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2\right) \\
&\geq \phi\left(\left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2, \left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2, ..., \left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2\right) \\
&\geq \left(\frac{\mu-\theta^{2k+1}}{\mu+\theta^{2k+1}}\right)^2 \geq \left(\frac{\mu-\theta^{2k+2}}{\mu+\theta^{2k+2}}\right)^2.
\end{aligned}$$

Thus

$$\alpha_{2k+2} \geq \left(\frac{\mu-\theta^{2k+2}}{\mu+\theta^{2k+2}}\right)^2 \tag{3}$$

Continuing in this way,we have

$$\alpha_n \geq \left(\frac{\mu-\theta^{n}}{\mu+\theta^{n}}\right)^2, n=1,2,3... \tag{4}$$

Now consider

$$\begin{aligned}
M\left(y_{2k+2n-1}, y_{2k+2n}, t\right) &\geq M\left(y_{2k+2n-1}, y_{2k+2n}, qt\right), \text{ since } q < 1 \text{ and } M(x,y,.) \text{ is increasing} \\
&= M\begin{pmatrix} S\left(x_{2n-1}, x_{2n}, x_{2n+1}, ...x_{2k+2n-3}, x_{2k+2n-2}\right), \\ T\left(x_{2n}, x_{2n+1}, x_{2n+2}, ...x_{2k+2n-2}, x_{2k+2n-1}\right), qt \end{pmatrix} \\
&\geq \phi\begin{pmatrix} M(gx_{2n-1}, fx_{2n}, t), M(fx_{2n}, gx_{2n+1}, t), \\ M(gx_{2n+1}, fx_{2n+2}, t), M(fx_{2n+2}, gx_{2n+3}, t), \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ M(gx_{2k+2n-3}, fx_{2k+2n-2}, t), M(fx_{2k+2n-2}, gx_{2k+2n-1}, t) \end{pmatrix} \\
&\geq \phi\left(\alpha_{2n-1}, \alpha_{2n}, \alpha_{2n+1}, ..., \alpha_{2k+2n-3}, \alpha_{2k+2n-2}\right), \text{ since } \phi \text{ and } M \text{ are increasing} \\
&\geq \phi\left(\left(\frac{\mu-\theta^{2n-1}}{\mu+\theta^{2n-1}}\right)^2, \left(\frac{\mu-\theta^{2n}}{\mu+\theta^{2n}}\right)^2, ..., \left(\frac{\mu-\theta^{2k+2n-2}}{\mu+\theta^{2k+2n-2}}\right)^2\right) \text{ from (4)} \\
&\geq \phi\left(\left(\frac{\mu-\theta^{2k+2n-2}}{\mu+\theta^{2k+2n-2}}\right)^2, \left(\frac{\mu-\theta^{2k+2n-2}}{\mu+\theta^{2k+2n-2}}\right)^2, ..., \left(\frac{\mu-\theta^{2k+2n-2}}{\mu+\theta^{2k+2n-2}}\right)^2\right) \\
&\geq \left(\frac{\mu-\theta^{2k+2n-2}}{\mu+\theta^{2k+2n-2}}\right)^2 \\
&\geq \left(\frac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2.
\end{aligned}$$

Thus

$$M(y_{2k+2n-1}, y_{2k+2n}, t) \geq \left(\frac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2 \tag{5}$$

Also

$$\begin{aligned}
M\left(y_{2k+2n}, y_{2k+2n+1}, t\right) &\geq M\left(y_{2k+2n}, y_{2k+2n+1}, qt\right), \text{ since } q < 1 \text{ and } \phi \text{ is increasing} \\
&= M\begin{pmatrix} S\left(x_{2n+1}, x_{2n+2}, x_{2n+3}, ..., x_{2k+2n-1}, x_{2k+2n}\right), \\ T\left(x_{2n}, x_{2n+1}, x_{2n+2}, ..., x_{2k+2n-2}, x_{2k+2n-1}\right), qt \end{pmatrix}, \\
&\geq \phi\begin{pmatrix} M(gx_{2n+1}, fx_{2n}, t), M(fx_{2n+2}, gx_{2n+1}, t), \\ M(gx_{2n+3}, fx_{2n+2}, t), M(fx_{2n+4}, gx_{2n+3}, t), \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ M(gx_{2k+2n-1}, fx_{2k+2n-2}, t), M(fx_{2k+2n}, gx_{2k+2n-1}, t) \end{pmatrix} \\
&\geq \phi\left(\alpha_{2n}, \alpha_{2n+1}, ..., \alpha_{2k+2n-2}, \alpha_{2k+2n-1}\right) \\
&\geq \phi\left(\left(\frac{\mu-\theta^{2n}}{\mu+\theta^{2n}}\right)^2, \left(\frac{\mu-\theta^{2n+1}}{\mu+\theta^{2n+1}}\right)^2, ..., \left(\frac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2\right) \text{ from (4)} \\
&\geq \phi\left(\left(\frac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2, \left(\frac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2, ..., \left(\frac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2\right) \\
&\geq \left(\frac{\mu-\theta^{2k+2n-1}}{\mu+\theta^{2k+2n-1}}\right)^2, \text{ since } \phi(t,t,t,...,t) \geq t \\
&\geq \left(\frac{\mu-\theta^{2k+2n}}{\mu+\theta^{2k+2n}}\right)^2.
\end{aligned}$$

Thus

$$M\left(y_{2k+2n}, y_{2k+2n+1}, t\right) \geq \left(\frac{\mu - \theta^{2k+2n}}{\mu + \theta^{2k+2n}}\right)^2 \tag{6}$$

Hence from (5) and (6)we have

$$M\left(y_{2k+n}, y_{2k+n+1}, t\right) \geq \left(\frac{\mu - \theta^{2k+n}}{\mu + \theta^{2k+n}}\right)^2 \quad for \quad n = 1, 2, ... \tag{7}$$

Now for $n, p \in N$, we have

$$\begin{aligned} M(y_{2k+n}, y_{2k+n+p}, t) &\geq M(y_{2k+n}, y_{2k+n+1}, \frac{t}{p}) * M(y_{2k+n+1}, y_{2k+n+2}, \frac{t}{p}) * ... * M\left(y_{2k+n+p-1}, y_{2k+n+p}, \frac{t}{p}\right) \\ &\geq \left(\frac{\mu - \theta^{2k+n}}{\mu + \theta^{2k+n}}\right)^2 * \left(\frac{\mu - \theta^{2k+n+1}}{\mu + \theta^{2k+n+1}}\right)^2 * .... * \left(\frac{\mu - \theta^{2k+n+p-1}}{\mu + \theta^{2k+n+p-1}}\right)^2, \text{from } (7) \\ &\to 1 * 1 * 1 * .... * 1 \text{ as } n \ \to \infty \\ &= 1. \end{aligned}$$

Hence $\{y_{2k+n}\}$ is a Cauchy sequence in X.
Since X is complete,there exists $z \in X$ such that $y_{2k+n} \to z$ as $n \to \infty$.
From(2.1.4),there exists $u \in X$ such that

$$z = fu = gu \tag{8}$$

Now consider

$$\begin{aligned} M\left(S\left(u, u, ..., u, u\right), y_{2k+2n}, qt\right) &= M\left(S\left(u, u, ..., u, u\right), T\left(x_{2n}, x_{2n+1}, ..., x_{2n+2k-2}, x_{2n+2k-1}\right), qt\right) \\ &\geq \phi\begin{pmatrix} M\left(gu, fx_{2n}, t\right), M\left(fu, gx_{2n+1}, t\right), \\ ...., \\ M\left(gu, fx_{2n+2k-2}, t\right), M\left(fu, gx_{2k+2n-1}, t\right) \end{pmatrix}. \end{aligned}$$

Letting $n \to \infty$ and using (8),we get

$$M(S(u, u, ..., u, u), fu, qt) \geq \phi(1, 1, ..., 1, 1) \geq 1$$

which implies that

$$S(u, u, ..., u, u) = fu \tag{9}$$

Similarly we can prove that

$$T(u, u, ..., u, u) = gu \tag{10}$$

Since (f, S) and (g, T) are jointly $2k$-weakly compatible pairs, we have

$$fz = f(fu) = f(S(u, u, ..., u)) = S(fu, fu, ..., fu) = S(z, z, ..., z) \tag{11}$$

and also

$$gz = T(z, z, ..., z, z) \tag{12}$$

Now consider

$$\begin{aligned} M(fz, z, qt) &= M(S(z, z, ..., z, z), T(u, u, ..., u, u), qt), \text{ from } (11), (8), (10) \\ &\geq \phi\begin{pmatrix} M(gz, fu, t), M(fz, gu, t), \\ M(gz, fu, t), M(fz, gu, t), \\ .... \\ M(gz, fu, t), M(fz, gu, t) \end{pmatrix} \\ &\geq \phi\begin{pmatrix} \min\{M(gz, z, t), M(fz, z, t)\}, \\ \min\{M\left(gz, z, t\right), M\left(fz, z, t\right)\}, \\ .... \\ \min\{M\left(gz, z, t\right), M\left(fz, z, t\right)\} \end{pmatrix} \\ &\geq \min\{M\left(gz, z, t\right), M\left(fz, z, t\right)\}. \end{aligned}$$

Thus

$$M(fz, z, qt) \geq \min\{M(gz, z, t), M(fz, z, t)\} \tag{13}$$

Similarly,we can show that

$$M(gz, z, qt) \geq \min\{M(z, fz, t), M(z, gz, t)\} \tag{14}$$

Thus from (13)and (14),we have

$$\min\{M(fz, z, qt), M(gz, z, qt)\} \geq \min\{M(z, fz, t), M(z, gz, t)\}$$

which inturn yields from condition(A) that

$$z = fz \quad and \quad z = gz \tag{15}$$

From (11),(12)and (15),we have

$$z = fz = gz = S(z, z, ..., z) = T(z, z, ..., z) \tag{16}$$

Suppose there exists $z' \in X$ such that

$$z' = fz' = gz' = S(z', z', ..., z', z') = T(z', z', ..., z', z')$$

Then from (2.1.2) we have

$$\begin{aligned} M(z, z', qt) &= M(S(z, z, ..., z, z), T(z', z', ..., z', z'), qt) \\ &\geq \phi\begin{pmatrix} M(gz, fz', t), M(fz, gz', t), \\ M(gz, fz', t), M(fz, gz', t), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ M(gz, fz', t), M(fz, gz', t) \end{pmatrix} \\ &= \phi(M(z, z', t), M(z, z', t), ..., M(z, z', t)) \\ &\geq M(z, z', t) \end{aligned}$$

From the condition(A), we have $z' = z$.
Thus z is the unique point in X satisfying (16). □

Now we give an example to illustrate our main Theorem 2.1.

Example 2.2. *Let $X = [0,1], a * b = ab, M(x, y, t) = e^{-\frac{|x-y|}{t}}$ and $k = 1$.Define $\phi : [0,1]^2 \to [0,1]$ as $\phi(x_1, x_2) = \min\{x_1, x_2\}$.Let $S, T : X^2 \to X$ and $f, g : X \to X$ be defined as $S(x, y) = \frac{3x^2+2y}{72}$, $T(x, y) = \frac{2x+3y^2}{72}$, $fx = \frac{x}{6}$ and $gx = \frac{x^2}{4}$. Now for $x_1, x_2, y_1, y_2 \in X$,we have*

$$\begin{aligned} |S(x_1, x_2) - T(y_1, y_2)| &= |\tfrac{3x_1^2+2x_2}{72} - \tfrac{2y_1+3y_2^2}{72}| \\ &= \tfrac{1}{72}|3x_1^2 - 2y_1 + 2x_2 - 3y_2^2| \\ &\leq \tfrac{1}{36}\max\{|3x_1^2 - 2y_1|, |2x_2 - 3y_2^2|\}. \end{aligned}$$

Now, we have

$$\begin{aligned} M(S(x_1, x_2), T(y_1, y_2), \tfrac{1}{3}t) &= e^{-\frac{|S(x_1,x_2)-T(y_1,y_2)|}{\frac{1}{3}t}} \\ &\geq e^{-\frac{1}{36}\frac{\max\{|3x_1^2-2y_1|,|2x_2-3y_2^2|\}}{\frac{t}{3}}} \\ &= e^{-\frac{\max\{|3x_1^2-2y_1|,|2x_2-3y_2^2|\}}{12t}} \\ &= e^{-\frac{\max\{|\frac{x_1^2}{4}-\frac{y_1}{6}|,|\frac{x_2}{6}-\frac{y_2^2}{4}|\}}{t}} \\ &\geq \min\left\{e^{-\frac{|\frac{x_1^2}{4}-\frac{y_1}{6}|}{t}}, e^{-\frac{|\frac{x_2}{6}-\frac{y_2^2}{4}|}{t}}\right\} \\ &= \min\{M(gx_1, fy_1, t), M(fx_2, gy_2, t)\} \\ &= \phi(M(gx_1, fy_1, t), M(fx_2, gy_2, t). \end{aligned}$$

Thus (2.1.2) is satisfied with $q = \frac{1}{3}$.
One can easily verify the remaining conditions of Theorem 2.1. Clearly 0 is the unique point in X satisfying (16).

Corollary 2.3. *Let $(X, M, *)$ be fuzzy metric space with the condition (A) and $S, T : X^{2k} \to X$ and $f : X \to X$ be mappings satisfying:*

(2.3.1) $S(X^{2k}) \subseteq f(X), T(X^{2k}) \subseteq f(X)$,

(2.3.2) $M(S(x_1, x_2, ..., x_{2k}), T(y_1, y_2, ..., y_{2k}), qt) \geq \phi\left(M(fx_1, fy_1, t), M(fx_2, fy_2, t), ..., M\left(fx_{2k}, fy_{2k}, t\right)\right)$ $\forall x_1, x_2, ..., x_{2k}, y_1, y_2, ..., y_{2k} \in X, \forall t > 0$ *and* $0 < q < 1$,

(2.3.3) *$f(X)$ is a complete subspace of X.*

(2.3.4) *Either (f, S) or (f, T) is a $2k$-weakly compatible pair. Then there exists a unique $u \in X$ such that $u = fu = S(u, u, ..., u, u) = T(u, u, ..., u, u)$.*

Corollary 2.4. *Let $(X, M, *)$ be a complete fuzzy metric space with the condition(A) and $S, T : X^{2k} \to X$ be mappings satisfying:*

(2.4.1) $M(S(x_1, x_2, ..., x_{2k}), T(y_1, y_2, ..., y_{2k}), qt) \geq \phi\left(M\left(x_1, y_1, t\right), M\left(x_2, y_2, t\right), ..., M\left(x_{2k}, y_{2k}, t\right)\right)$ $\forall x_1, x_2, ..., x_{2k}, y_1, y_2, ..., y_{2k} \in X, \forall t > 0$ *and* $0 < q < 1$.

Then there exists a unique $u \in X$ such that $u = S(u, u, ..., u) = T(u, u, ..., u)$.

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UNIQUE COMMON FIXED POINT THEOREM FOR FOUR MAPS IN COMPLEX VALUED $S-$ METRIC SPACES

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Abstract. In this paper we obtain a common fixed point theorem for the two weakly compatible pairs of mappings satisfying a contractive condition in complex valued S-metric spaces.

1. Introduction

It is a well-known fact that the mathematical results regarding fixed points of contraction type mappings are very useful for determining the existence and uniqueness of solutions to various mathematical models. Over the last 40 years, the theory of fixed points has been developed regarding the results that are related to finding the fixed points of self and nonself nonlinear mappings in a metric space.

Several authors proved fixed point results in different types of generalized metric spaces.

Azam et al. [**2**] introduced the concept of a complex valued metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions. Later several authors proved fixed and common fixed point theorems in complex valued metric spaces, for example, refer [**1, 2, 3, 5, 13, 9, 11, 12, 14, 15**].

Throughout this paper, let **C** denote the set of all complex numbers.

A Complex number $z \in \mathbf{C}$ is an ordered pair of real numbers, whose first co-ordinate is called $Re(z)$ and second co-ordinate is called $Im(z)$. Let $z_1, z_2 \in \mathbf{C}$.

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Define a partial order $\precsim$ on $\mathbf{C}$ follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leqslant Re(z_2),\ Im(z_1) \leqslant Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the following holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Azam [**2**] defined the complex metric as follows:

Definition 1.1. ([**2**]) Let X be a non-empty set. A function $d : X \times X \to \mathbf{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \precsim d(x, z) + d(z, y)$.

The pair (X, d) is called a complex valued metric space.

Sedghi et al. [**16**] introduced the concept of $S-$ metric space as follows.

Definition 1.2. ([**16**]) Let X be a non-empty set. A $S-$metric on X is a function $S : X^3 \longrightarrow [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

(S1) $S(x, y, z) = o$ if and only if $x = y = z$,
(S2) $S(x, y, z) \leqslant S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an $S-$metric space.

Following examples of $S-$metric space are due to[**16**].

Example 1.1. 1) Let $X = \mathbf{R}^n$ and $\|.\|$ a norm on X. Then

$$S(x, y, z) = \|yz - 2x\| + \|x + y\|$$

is an $S-$metric space.

2) Let $X = \mathbf{R}^n$ and $\|.\|$ a norm on X. Then

$$S(x, y, z) = \|x - z\| + \|y - z\|$$

is an $S-$metric space.

Later some authors proved fixed point results in $S-$metric spaces, for example [**4, 6, 8, 10, 16**].

Lemma 1.1 ([**16**]). *Let (X, S) be a $S-$metric space. If there exist $\{x_n\}$ and $\{y_n\}$ such that*

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y,$$

then

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

For $\{y_n\} = y$ the above lemma becomes as follows.

Lemma 1.2. *Let (X, S) be a $S-$metric space. If there exists $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} S(x_n, x_n, y) = S(x, x, y)$.*

Nabil et al. [**7**] introduced the concept of complex valued $S-$ metric space as follows.

Definition 1.3. ([**7**]) Let X be a non-empty set. A complex valued S-metric on X is a function $S : X^3 \to \mathbf{C}$ that satisfies the following conditions, for all $x, y, z, a \in X$:
(i) $0 \precsim S(x, y, z)$,
(ii) $S(x, y, z) = 0$ if and only if $x = y = z$,
(iii) $S(x, y, z) \precsim S(x, x, a) + S(y, y, a) + S(z, z, a)$.
The pair (X, S) is called a complex valued S-metric space.

Example 1.2. Let X = $\mathbf{C}$. Define $S : \mathbf{C}^3 \to \mathbf{C}$ by:
$S(z_1, z_2, z_3) =$
$[|Re(z_1) - Re(z_3)| + |Re(z_2) - Re(z_3)|] + i[|Im(z_1) - Im(z_3)| + |Im(z_2) - Im(z_3)|]$.

Then (X, S) is a complex valued S-metric space.

Definition 1.4. ([**7**]) If (X, S) is called a complex valued S-metric space, then

(1) A sequence $\{x_n\}$ in X converges to x if and only if for all ϵ such that $0 \prec \epsilon \in \mathbf{C}$, there exists a natural number n_0 such that for all $n \geqslant n_0$, we have $S(x_n, x_n, x) \prec \epsilon$ and we denote this by $\lim_{n\to\infty} x_n = x$.
(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for all ϵ such that $0 \prec \epsilon \in \mathbf{C}$, there exists a natural number n_0 such that for all $n, m \geqslant n_0$, we have $S(x_n, x_n, x_m) \prec \epsilon$.
(3) An S-metric space (X, S) is said to be complete if for every Cauchy sequence is convergent.

Lemma 1.3 ([**7**]). *Let (X, S) be a complex valued S-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|S(x_n, x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.4 ([**7**]). *Let (X, S) be a complex valued S-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|S(x_n, x_n, x_{n+m})| \to 0$ as $n \to \infty$ and $m \to \infty$.*

Lemma 1.5 ([**7**]). *Let (X, S) be a complex valued S-metric space. Then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

2. Main results

Recently Naval Singh et al. [**13**] proved the following theorem in complex valued metric spaces as follows.

Theorem 2.1. *Let (X, d) be a complete complex valued metric space and $S, T : X \to X$. If there exist mappings $\lambda, \mu, \gamma, \delta : X \times X \times X \to [0, 1)$ such that for all $x, y \in X$, the following is valid*
(a)

$$\lambda(TSx, y, a) \leqslant \lambda(x, y, a) \text{ and } \lambda(x, STy, a) \leqslant \lambda(x, y, a),$$
$$\mu(TSx, y, a) \leqslant \mu(x, y, a) \text{ and } \mu(x, STy, a) \leqslant \mu(x, y, a),$$
$$\gamma(TSx, y, a) \leqslant \gamma(x, y, a) \text{ and } \gamma(x, STy, a) \leqslant \gamma(x, y, a),$$
$$\delta(TSx, y, a) \leqslant \delta(x, y, a) \text{ and } \delta(x, STy, a) \leqslant \delta(x, y, a);$$

(b)

$$d(Sx, Ty) \precsim \lambda(x, y, a)d(x, y) + \mu(x, y, a)\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + \gamma(x, y, a)\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)}$$
$$+\delta(x, y, a)\frac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{1+d(x,Ty)+d(y,Sx)};$$

(c) $\lambda(x, y, a) + \mu(x, y, a) + \gamma(x, y, a) + \delta(x, y, a) < 1$,

then S and T have a unique common fixed point.

In this paper we generalize the Theorem (2.1) in complex valued $S-$metric spaces for four maps satisfying more general contractive condition. First we prove a proposition which is needed to prove our main Theorem.

Proposition 2.1. *Let (X, S) be a complex valued S-metric space and $F, G, f, g : X \to X$. Let $y_0 \in X$ and define the sequence $\{y_n\}$ by*

$$y_{2n+1} = gx_{2n+1} = Fx_{2n};\ y_{2n+2} = fx_{2n+2} = Gx_{2n+1}.\text{for all} n = 0, 1, 2....$$

Assume that there exists a mapping $\lambda_1 : X \times X \times X \to [0, 1)$ such that
(i) $\lambda_1(Fx, y, a) \leqslant \lambda_1(fx, y, a)$ *and* $\lambda_1(x, Gy, a) \leqslant \lambda_1(x, gy, a)$,
(ii) $\lambda_1(Gx, y, a) \leqslant \lambda_1(gx, y, a)$ *and* $\lambda_1(x, Fy, a) \leqslant \lambda_1(x, fy, a)$.
fora ll $x, y \in X$ and for a fixed element $a \in X$ and $n = 0, 1, 2, ...$ Then

$$\lambda_1(y_{2n}, y, a) \leqslant \lambda_1(y_0, y, a) \text{ and } \lambda_1(x, y_{2n+1}, a) \leqslant \lambda_1(x, y_1, a), \text{ for all} x, y \in X$$

.

Proof. Let $x, y \in X$ and $n = 0, 1, 2....$ Then we have

$$\lambda_1(y_{2n}, y, a) = \lambda_1(Gx_{2n-1}, y, a) \leqslant \lambda_1(gx_{2n-1}, y, a) = \lambda(y_{2n-1}, y, a) =$$
$$\lambda_1(Fx_{2n-2}, y, a) \leqslant \lambda_1(fx_{2n-2}, y, a) = \lambda(y_{2n-2}, y, a) = \lambda_1(Gx_{2n-3}, y, a)$$
$$\leqslant \lambda_1(gx_{2n-3}, y, a) = \lambda_1(y_{2n-3}, y, a) \cdots = \lambda_1(y_0, y, a).$$

Thus $\lambda_1(y_{2n}, y, a) \leqslant \lambda_1(y_0, y, a)$.
Similarly we have

$$\lambda_1(x, y_{2n+1}, a) = \lambda_1(x, Fx_{2n}, a) \leqslant$$
$$\lambda_1(x, fx_{2n}, a) = \lambda_1(x, y_{2n}, a) = \lambda_1(x, Gx_{2n-1}, a)$$
$$\leqslant \lambda_1(x, gx_{2n-1}, a) = \lambda_1(x, y_{2n-1}, a) = \lambda_1(x, Fx_{2n-2}, a) \leqslant \lambda_1(x, fx_{2n-2}, a) =$$
$$\lambda_1(x, y_{2n-2}, a) \cdots = \lambda_1(x, y_1, a).$$

Thus $\lambda_1(x, y_{2n+1}, a) \leqslant \lambda_1(x, y_1, a)$. □

Theorem 2.2. *Let (X, S) be a complex valued $S-$metric space and $F, G, f, g : X \to X$ satisfying the conditions .*
(2.2.1) $GX \subseteq fX$ *and* $FX \subseteq gX$,
(2.2.2) *The pairs (F, f) and (G, g) are weakly compatible ,*

(2.2.3) *fX or gX is a complete subspace of X,*

(2.2.4) *If there exist mappings $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 : X \times X \times X \to [0, 1)$ such that $\lambda_n(Fx, y, a) \leqslant \lambda_n(fx, y, a)$; $\lambda_n(Gx, y, a) \leqslant \lambda_n(gx, y, a)$ and $\lambda_n(x, Fy, a) \leqslant \lambda_n(x, fy, a)$; $\lambda_n(x, Gy, a) \leqslant \lambda_n(x, gy, a), \forall n = 1, 2, 3..., 7,$ for all $x, y \in X$ and for a fixed element $a \in X$,*

(2.2.5)

$$\begin{aligned} S(Fx, Fx, Gy) \precsim\ & \lambda_1(fx, gy, a)S(fx, fx, gy) + \lambda_2(fx, gy, a)S(fx, fx, Fx) \\ & +\lambda_3(fx, gy, a)S(gy, gy, Gy) \\ & +\lambda_4(fx, gy, a)[S(gy, gy, Fx) + S(fx, fx, Gy)] \\ & +\lambda_5(fx, gy, a)\left(\frac{S(fx,fx,Fx)S(gy,gy,Gy)}{1+S(fx,fx,gy)}\right) \\ & +\lambda_6(fx, gy, a)\left(\frac{S(gy,gy,Fx)S(fx,fx,Gy)}{1+S(fx,fx,gy)}\right) \\ & +\lambda_7(fx, gy, a)\left(\frac{S(fx,fx,Fx)S(fx,fx,Gy)+S(gy,gy,Gy)S(gy,gy,Fx)}{1+S(fx,fx,Gy)+S(gy,gy,Fx)}\right) \end{aligned}$$

for all $x, y \in X$ and for a fixed element $a \in X$, where

(2.2.6) $(\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + \lambda_7)(x, y, a) < 1.$

Then F, G, f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. We define a sequence $\{y_n\}$ in X such that $y_{2n+1} = gx_{2n+1} = Fx_{2n}$ and $y_{2n+2} = fx_{2n+2} = Gx_{2n+1}, n = 0, 1, 2, ...$ From(2.2.5) we have

$$\begin{aligned} & S(y_{2n+1}, y_{2n+1}, y_{2n+2}) = S(Fx_{2n}, Fx_{2n}, Gx_{2n+1}) \\ & \precsim \lambda_1(y_{2n}, y_{2n+1}, a)S(y_{2n}, y_{2n}, y_{2n+1}) + \lambda_2(y_{2n}, y_{2n+1}, a)S(y_{2n}, y_{2n}, y_{2n+1}) \\ & \quad +\lambda_3(y_{2n}, y_{2n+1}, a)S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \\ & \quad +\lambda_4(y_{2n}, y_{2n+1}, a)[S(y_{2n+1}, y_{2n+1}, y_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+2})] \\ & \quad +\lambda_5(y_{2n}, y_{2n+1}, a)\left(\frac{S(y_{2n},y_{2n},y_{2n+1})S(y_{2n+1},y_{2n+1},y_{2n+2})}{1+S(y_{2n},y_{2n},y_{2n+1})}\right) \\ & \quad +\lambda_6(y_{2n}, y_{2n+1}, a)\left(\frac{S(y_{2n+1},y_{2n+1},y_{2n+1})S(y_{2n},y_{2n},y_{2n+1})}{1+S(y_{2n},y_{2n},y_{2n+1})}\right) \\ & \quad +\lambda_7(y_{2n}, y_{2n+1}, a)\left(\frac{\begin{array}{l} S(y_{2n}, y_{2n}, y_{2n+1})S(y_{2n}, y_{2n}, y_{2n+2}) \\ +S(y_{2n+1}, y_{2n+1}, y_{2n+2})S(y_{2n+1}, y_{2n+1}, y_{2n+1}) \end{array}}{1+S(y_{2n},y_{2n},y_{2n+1})+S(y_{2n+1},y_{2n+1},y_{2n+1})}\right) \end{aligned}$$

Since $S(x, x, x) = 0$, we have

$$\begin{aligned} & |S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \\ & \leqslant \lambda_1(y_{2n}, y_{2n+1}, a)\,|S(y_{2n}, y_{2n}, y_{2n+1})| \\ & \quad +\lambda_2(y_{2n}, y_{2n+1}, a)\,|S(y_{2n}, y_{2n}, y_{2n+1})| \\ & \quad +\lambda_3(y_{2n}, y_{2n+1}, a)\,|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \\ & \quad +\lambda_4(y_{2n}, y_{2n+1}, a)\,|S(y_{2n}, y_{2n}, y_{2n+1})| \\ & \quad +\lambda_4(y_{2n}, y_{2n+1}, a)\,|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \\ & \quad +\lambda_5(y_{2n}, y_{2n+1}, a)\,|S(y_{2n+1}, y_{2n+1}, y_{2n+2})|\left|\frac{S(y_{2n},y_{2n},y_{2n+1})}{1+S(y_{2n},y_{2n},y_{2n+1})}\right| \\ & \quad +\lambda_7(y_{2n}, y_{2n+1}, a)\,|S(y_{2n}, y_{2n}, y_{2n+1})|\left|\frac{S(y_{2n},y_{2n},y_{2n+2})}{1+S(y_{2n},y_{2n},y_{2n+2})}\right|. \end{aligned}$$

$$\begin{aligned}&\left|S(y_{2n+1},y_{2n+1},y_{2n+2})\right|\\&\quad\leqslant(\lambda_1+\lambda_2+\lambda_4+\lambda_7)(y_{2n},y_{2n+1},a)\left|S(y_{2n},y_{2n},y_{2n+1})\right|\\&\qquad\quad+(\lambda_3+\lambda_4+\lambda_5)(y_{2n},y_{2n+1},a)\left|S(y_{2n+1},y_{2n+1},y_{2n+2})\right|.\end{aligned}$$

Using Proposition (2.1), we get

$$\begin{aligned}\left|S(y_{2n+1},y_{2n+1},y_{2n+2})\right|\leqslant&(\lambda_1+\lambda_2+\lambda_4+\lambda_7)(y_0,y_1,a)\left|S(y_{2n},y_{2n},y_{2n+1})\right|\\&+(\lambda_3+\lambda_4+\lambda_5)(y_0,y_1,a)\left|S(y_{2n+1},y_{2n+1},y_{2n+2})\right|\end{aligned}$$

which in turn implies that

$$\left|S(y_{2n+1},y_{2n+1},y_{2n+2})\right|\leqslant\left(\frac{(\lambda_1+\lambda_2+\lambda_4+\lambda_7)(y_0,y_1,a)}{1-(\lambda_3+\lambda_4+\lambda_5)(y_0,y_1,a)}\right)\left|S(y_{2n},y_{2n},y_{2n+1})\right|.$$

Let $h_1=\left(\frac{(\lambda_1+\lambda_2+\lambda_4+\lambda_7)(y_0,y_1,a)}{1-(\lambda_3+\lambda_4+\lambda_5)(y_0,y_1,a)}\right)$. Thus

$$\left|S(y_{2n+1},y_{2n+1},y_{2n+2})\right|\leqslant h_1\left|S(y_{2n},y_{2n},y_{2n+1})\right|. \text{ ....(1)}$$

Similarly using $S(x,y,y)=S(x,x,y)$ and proceeding as above we can show that

$$\left|S(y_{2n+2},y_{2n+2},y_{2n+3})\right|\leqslant h_2\left|S(y_{2n+1},y_{2n+1},y_{2n+2})\right| \text{........(2)}$$

where $h_2=\left(\frac{(\lambda_1+\lambda_3+\lambda_4+\lambda_7)(y_0,y_1,a)}{1-(\lambda_2+\lambda_4+\lambda_5)(y_0,y_1,a)}\right)$.

Let $h=max\{h_1,h_2\}$, then $0\leqslant h<1$, since $h_1,h_2\in[0,1)$. Hence from (1) and (2), we have $\left|S(y_n,y_n,y_{n+1})\right|\leqslant h\left|S(y_{n-1},y_{n-1},y_n)\right|, n=1,2,3,...$ Repeated use of above inequality gives

$$\begin{aligned}\left|S(y_k,y_k,y_{k+1})\right|&\leqslant h^k\left|S(y_0,y_0,y_1)\right| \text{ ....(3)}\\&\to 0\quad as\quad k\to\infty \text{ ....(4)}\end{aligned}$$

Hence for any $m>n$, we have

$$\begin{aligned}\left|S(y_n,y_n,y_m)\right|&=2\left[\begin{array}{l}\left|S(y_n,y_n,y_{n+1})\right|+\left|S(y_{n+1},y_{n+1},y_{n+2})\right|+\\ ...+\left|S(y_{m-1},y_{m-1},y_m)\right|\end{array}\right]\\&=2(h^n+h^{n+1}+...+h^{m-1})\left|S(y_0,y_0,y_1)\right| \text{ from (3)}\\&\leqslant\frac{2h^n}{1-h}\left|S(y_0,y_0,y_1)\right|\end{aligned}$$

and

$$\left|S(y_n,y_n,y_m)\right|\leqslant\frac{2h^n}{1-h}\left|S(y_0,y_0,y_1)\right|\to 0 \text{ as } m,n\to\infty.$$

Hence $\{y_n\}$ is a Cauchy sequence in X.

Now suppose fX is a complete subspace of X. Since $y_{2n+2}=fx_{2n+2}\in f(X)$ and $\{y_n\}$ is a Cauchy sequence, there exists $z\in f(X)$ such that $y_{2n+2}\to z$ as $n\to\infty$. Then there exists $u\in X$ such that $fu=z$. Thus

$$\lim_{n\to\infty}Fx_{2n}=\lim_{n\to\infty}gx_{2n+1}=\lim_{n\to\infty}Gx_{2n+1}=\lim_{n\to\infty}fx_{2n+2}=z.$$

Consider

$$\begin{aligned}
&S(Fu,Fu,Gx_{2n+1})\\
&\precsim \lambda_1(fu,y_{2n+1},a)S(fu,fu,y_{2n+1})\\
&\quad+\lambda_2(fu,y_{2n+1},a)S(fu,fu,Fu)\\
&\quad+\lambda_3(fu,y_{2n+1},a)S(y_{2n+1},y_{2n+1},y_{2n+2})\\
&\quad+\lambda_4(fu,y_{2n+1},a)[S(y_{2n+1},y_{2n+1},Fu)+S(fu,fu,y_{2n+2})]\\
&\quad+\lambda_5(fu,y_{2n+1},a)\left(\frac{S(fu,fu,Fu)S(y_{2n+1},y_{2n+1},y_{2n+2})}{1+S(fu,fu,y_{2n+1})}\right)\\
&\quad+\lambda_6(fu,y_{2n+1},a)\left(\frac{S(y_{2n+1},y_{2n+1},Fu)S(fu,fu,y_{2n+2})}{1+S(fu,fu,y_{2n+1})}\right)\\
&+\lambda_7(fu,y_{2n+1},a)\left(\frac{S(fu,fu,Fu)S(fu,fu,y_{2n+2})+S(y_{2n+1},y_{2n+1},y_{2n+2})S(y_{2n+1},y_{2n+1},Fu)}{1+S(fu,fu,y_{2n+2})+S(y_{2n+1},y_{2n+1},Fu)}\right)
\end{aligned}$$

$$\begin{aligned}
&|S(Fu,Fu,Gx_{2n+1})|\\
&\leqslant \lambda_1(fu,y_{2n+1},a)|S(fu,fu,y_{2n+1})|\\
&\quad+\lambda_2(fu,y_{2n+1},a)|S(fu,fu,Fu)|\\
&\quad+\lambda_3(fu,y_{2n+1},a)|S(y_{2n+1},y_{2n+1},y_{2n+2})|\\
&\quad+\lambda_4(fu,y_{2n+1},a)|S(y_{2n+1},y_{2n+1},Fu)+S(fu,fu,y_{2n+2})|\\
&\quad+\lambda_5(fu,y_{2n+1},a)\left(\frac{|S(fu,fu,Fu)||S(y_{2n+1},y_{2n+1},y_{2n+2})|}{|1+S(fu,fu,y_{2n+1})|}\right)\\
&\quad+\lambda_6(fu,y_{2n+1},a)\left(\frac{|S(y_{2n+1},y_{2n+1},Fu)||S(fu,fu,y_{2n+2})|}{|1+S(fu,fu,y_{2n+1})|}\right)\\
&\quad+\lambda_7(fu,y_{2n+1},a)\left(\frac{\begin{array}{c}|S(fu,fu,Fu)||S(fu,fu,y_{2n+2})|\\ +|S(y_{2n+1},y_{2n+1},y_{2n+2})||S(y_{2n+1},y_{2n+1},Fu)\end{array}}{|1+S(fu,fu,y_{2n+1})+S(y_{2n+1},y_{2n+1},Fu)|}\right)
\end{aligned}$$

Letting $n \longrightarrow \infty$ and using Lemma 1.2 and 1.5, we get

$$|S(Fu,Fu,z)| \leqslant \lambda_2(z,z,a)\,|S(z,z,Fu)| + \lambda_4(z,z,a)\,|S(z,z,Fu)|$$

from(4), Lemma 1.3 $(1-(\lambda_2+\lambda_4)(z,z,a))\,|S(z,z,Fu)| \leqslant 0$ which in turn yields from(2.2.6) that $|S(Fu,Fu,z)| \leqslant 0$. Therefore $|S(Fu,Fu,z)| = 0$. Hence $Fu = z$. Thus $fu = Fu = z$. Since $FX \subseteq gX$, there exists $v \in X$ such that $Fu = gv$. Thus $fu = Fu = gv = z$. Again from (2.2.5), we have

$$\begin{aligned}
|S(z,z,Gv)| = |S(Fu,Fu,Gv)|\\
&\leqslant \lambda_1(fu,gv,a)|S(fu,fu,gv)| + \lambda_2(fu,gv,a)|S(fu,fu,Fu)|\\
&\quad+\lambda_3(fu,gv,a)|S(gv,gv,Gv)|\\
&\quad+\lambda_4(fu,gv,a)|S(gv,gv,Fu)+S(fu,fu,Gv)|\\
&\quad+\lambda_5(fu,gv,a)\left(\frac{|S(fu,fu,Fu)||S(gv,gv,Gv)|}{|1+S(fu,fu,gv)|}\right)\\
&\quad+\lambda_6(fu,gv,a)\left(\frac{|S(gv,gv,Fu)||S(fu,fu,Gv)|}{|1+S(fu,fu,gv)|}\right)\\
&\quad+\lambda_7(fu,gv,a)\left(\frac{|S(fu,fu,Fu)||S(fu,fu,Gv)|+|S(gv,gv,Gv)||S(gv,gv,Fu)|}{|1+S(fu,fu,Gv)+S(gv,gv,Fu)|}\right)
\end{aligned}$$

so that

$$|S(z,z,Gv)| \leqslant \lambda_3(z,z,a)\,|S(z,z,Gv)| + \lambda_4(z,z,a)\,|S(z,z,Gv)|.$$
$$(1-(\lambda_3+\lambda_4)(z,z,a))\,|S(z,z,Gv)| \leqslant 0$$

which in turn yields from(2.2.6) that $|S(z,z,Gv)| \leqslant 0$. Therefore $|S(z,z,Gv)| = 0$. Hence $Gv = z$. Thus

$$Gv = z = fu = Fu = gv. \text{......}(5)$$

Since (F, f) is weakly compatible, we have

$$fz = fFu = Ffu = Fz. \text{.........}(6)$$

$$\begin{aligned}
S(Fz,Fz,z) = S(Fz,Fz,Gv)& \\
\precsim\ & \lambda_1(fz,gv,a)S(fz,fz,gv) + \lambda_2(fz,gv,a)S(fz,fz,Fz) \\
& +\lambda_3(fz,gv,a)S(gv,gv,Gv) \\
& +\lambda_4(fz,gv,a)[S(gv,gv,Fz) + S(fz,fz,Gv)] \\
& +\lambda_5(fz,gv,a)\left(\tfrac{S(fz,fz,Fz)S(gv,gv,Gv)}{1+S(fz,fz,Gv)}\right) \\
& +\lambda_6(fz,gv,a)\left(\tfrac{S(gv,gv,Fz)S(fz,fz,Gv)}{1+S(fz,fz,Gv)}\right) \\
& +\lambda_7(fz,gv,a)\left(\tfrac{S(fz,fz,Fz)S(fz,fz,Gv)+S(gv,gv,Gv)S(gv,gv,Fz)}{1+S(fz,fz,Gv)+S(gv,gv,Fz)}\right) \\
=\ & \lambda_1(Fz,z,a)S(Fz,Fz,z) + \lambda_4(Fz,z,a)[S(z,z,Fz) + S(Fz,Fz,z)] \\
& +\lambda_6(Fz,z,a)\left(\tfrac{S(z,z,Fz)S(Fz,Fz,z)}{1+S(Fz,Fz,z)}\right) from(5)\, and\,(6)
\end{aligned}$$

$$\begin{aligned}
|S(Fz,Fz,z)| \leqslant\ & \lambda_1(Fz,z,a)\,|S(Fz,Fz,z)| \\
& +\lambda_4(Fz,z,a)\,|S(z,z,Fz) + S(Fz,Fz,z)| \\
& +\lambda_6(Fz,z,a)\,|S(z,z,Fz)|\left|\tfrac{S(Fz,Fz,z)}{1+S(Fz,Fz,z)}\right|.
\end{aligned}$$

$(1-(\lambda_1+2\lambda_4+\lambda_6)(Fz,z,a))\,|S(Fz,Fz,z)| \leqslant 0$
which in turn yields from (2.2.6) that $|S(Fz,Fz,z)| \leqslant 0$. Therefore $|S(Fz,Fz,z)| = 0$. Hence $Fz = z$. Thus

$$z = Fz = fz. \text{.........}(7)$$

Since the pair (G, g) is weakly compatible, we have

$$gz = gGv = Ggv = Gz. \text{......}(8)$$

From (2.2.5)

$$\begin{aligned}
S(z,z,Gz) = S(Fz,Fz,Gz)& \\
\precsim\ & \lambda_1(fz,gz,a)S(fz,fz,gz) + \lambda_2(fz,gz,a)S(fz,fz,Fz) \\
& +\lambda_3(fz,gz,a)S(gz,gz,Gz) \\
& +\lambda_4(fz,gz,a)[S(gz,gz,Fz) + S(fz,fz,Gz)] \\
& +\lambda_5(fz,gz,a)\left(\tfrac{S(fz,fz,FZ)S(gz,gz,Gz)}{1+S(fz,fz,gz)}\right) \\
& +\lambda_6(fz,gz,a)\left(\tfrac{S(gz,gz.Fz)S(fz,fz,Gz)}{1+S(fz,fz,gz)}\right) \\
& +\lambda_7(fz,gz,a)\left(\tfrac{S(fz,fz,Fz)S(fz,fz,Gz)+S(gz,gz,Gz)S(gz,gz,Fz)}{1+S(fz,fz,Gz)+S(gz,gz,Fz)}\right)
\end{aligned}$$

$$\begin{aligned}
|S(z,z,Gz)| \leqslant\ & \lambda_1(z,Gz,a)\,|S(z,z,Gz)| \\
& +\lambda_4(z,Gz,a)\,|S(Gz,Gz,z) + S(z,z,Gz)| \\
& +\lambda_6(z,Gz,a)\,|S(Gz,Gz,z)|\left|\tfrac{S(z,z,Gz)}{1+S(z,z,Gz)}\right| \text{ from (7), (8)}
\end{aligned}$$

$(1-(\lambda_1+2\lambda_4+\lambda_6)(z,Gz,a))\,|S(z,z,Gz)| \leqslant 0$ which in turn yields from (2.2.6) that $|S(z,z,Gz)| \leqslant 0$. Therefore $|S(z,z,Gz)| = 0$. Hence $Gz = z$, so that

$$Gz = gz = z. \;..............(9)$$

Thus from (7) and (9), z is a common fixed point of F, G, f and g. For uniqueness, let $z^* \in X$ be such that $fz^* = Fz^* = z^* = gz^* = Gz^*$.

Now from (2.2.5)

$$\begin{aligned} S(z,z,z^*) &= S(Fz,Fz,Gz^*) \\ &\precsim \lambda_1(fz,gz^*,a)S(fz,fz,gz^*) + \lambda_2(fz,gz^*,a)S(fz,fz,Fz) \\ &\quad + \lambda_3(fz,gz^*,a)S(gz^*,gz^*,Gz^*) \\ &\quad + \lambda_4(fz,gz^*,a)[S(gz^*,gz^*,Fz)+S(fz,fz,Gz^*)] \\ &\quad + \lambda_5(fz,gz^*,a)\left(\frac{S(fz,fz,Fz)S(gz^*,gz^*,Gz^*)}{1+S(fz,fz,gz^*)}\right) \\ &\quad + \lambda_6(fz,gz^*,a)\left(\frac{S(gz^*,gz^*,Fz)S(fz,fz,Gz^*)}{1+S(fz,fz,gz^*)}\right) \\ &\quad + \lambda_7(fz,gz^*,a)\left(\frac{S(fz,fz,Fz)S(fz,fz,Gz^*)+S(gz^*,gz^*,Gz^*)S(gz^*,gz^*,Fz)}{1+S(fz,fz,Gz^*)+S(gz^*,gz^*,Fz)}\right). \end{aligned}$$

$$\begin{aligned} |S(z,z,z^*)| &\leqslant \lambda_1(z,z^*,a)\,|S(z,z,z^*)| + \lambda_4(z,z^*,a)\,|S(z^*,z^*,z)+S(z,z,z^*)| \\ &\quad + \lambda_6(z,z^*,a)\,|S(z^*,z^*,z)| \left|\frac{S(z,z,z^*)}{1+S(z,z,z^*)}\right|. \end{aligned}$$

$|S(z,z,z^*)| \leqslant (\lambda_1+2\lambda_4+\lambda_6)(z,z^*,a)\,|S(z,z,z^*)|$.
$(1-(\lambda_1+2\lambda_4+\lambda_6)(z,z^*,a))\,|S(z,z,z^*)| \leqslant 0$ which in turn yields from (2.2.6) that $|S(z,z,z^*)| \leqslant 0$. Therfore $|S(z,z,z^*)| = 0$. Thus $z = z^*$.
Hence z is the unique common fixed point of F, G, f and g. Similarly we can prove the theorem if gX is a complete subspace of X. □

Now wegive an example to illustrate our main Theorem 2.2.

EXAMPLE 2.1. Let $X = [0,1]$ and $S : X \times X \times X \to C$ be defined by $S(x,y,z) = |x-z| + i|y-z|$. Then (X,S) is a complex valued $S-$ metric space. Define F, G, f and $g : X \to X$ by $Fx = \frac{x}{16}, Gx = \frac{x}{12}, fx = \frac{x}{4}$ and $gx = \frac{x}{3}$, for all $x \in X$. For fixed element $a = \frac{1}{3}$, define $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 : X \times X \times X \to [0,1]$ by

$$\lambda_1(x,y,a) = (\tfrac{x}{40}+\tfrac{y}{50}+a), \lambda_2(x,y,a) = \tfrac{xya}{10}, \lambda_3(x,y,a) = \tfrac{x^2y^2a^2}{10}, \lambda_4(x,y,a) = \tfrac{x^3y^3a^3}{10}, \lambda_5(x,y,a) = \tfrac{x^3+y^3+a^3}{10}, \lambda_6(x,y,a) = \tfrac{x^2ya^3}{50}, \lambda_7(x,y,a) = \tfrac{xy^3a^2}{40},$$

for all $x, y \in X$. Then
$\lambda_1(x,y,a)+\lambda_2(x,y,a)+\lambda_3(x,y,a)+2\lambda_4(x,y,a)+\lambda_5(x,y,a)+\lambda_6(x,y,a)+\lambda_7(x,y,a)$
$= (\frac{x}{40}+\frac{y}{50}+a) + \frac{xya}{10} + \frac{x^2y^2a^2}{10} + 2(\frac{x^3y^3a^3}{10}) + \frac{x^3+y^3+a^3}{10} + \frac{x^2ya^3}{50} + \frac{xy^3a^2}{40}$
$\leqslant (\frac{1}{40}+\frac{1}{50}+\frac{1}{3}) + \frac{1}{30} + \frac{1}{90} + \frac{2}{270} + \frac{55}{270} + \frac{1}{1350} + \frac{1}{360}$
$= \frac{3442}{5400} < 1.$
Hence $(\lambda_1+\lambda_2+\lambda_3+2\lambda_4+\lambda_5+\lambda_6+\lambda_7)(x,y,a) < 1$. We have

$$\lambda_1(Fx,y,a) = \lambda_1(\frac{x}{16},y,a) = (\frac{x}{640}+\frac{y}{50}+a)$$

$$\lambda_1(fx,y,a) = \lambda_1(\frac{x}{4},y,a) = (\frac{x}{160}+\frac{y}{50}+a).$$

Clearly $\lambda_1(Fx, y, a) \leqslant \lambda_1(fx, y, a)$. We have

$$\lambda_1(x, Fy, a) = \lambda_1(x, \frac{y}{16}, a) = (\frac{x}{40} + \frac{y}{800} + a)$$
$$\lambda_1(x, fy, a) = \lambda_1(x, \frac{y}{4}, a) = (\frac{x}{40} + \frac{y}{200} + a).$$

Clearly $\lambda_1(x, Fy, a) \leqslant \lambda_1(x, fy, a)$. We have

$$\lambda_1(Gx, y, a) = \lambda_1(\frac{x}{12}, y, a) = (\frac{x}{480} + \frac{y}{50} + a)$$
$$\lambda_1(gx, y, a) = \lambda_1(\frac{x}{3}, y, a) = (\frac{x}{120} + \frac{y}{50} + a).$$

Clearly $\lambda_1(Gx, y, a) \leqslant \lambda_1(gx, y, a)$. We have

$$\lambda_1(x, Gy, a) = \lambda_1(x, \frac{y}{12}, a) = (\frac{x}{40} + \frac{y}{600} + a)$$
$$\lambda_1(x, gy, a) = \lambda_1(x, \frac{y}{3}, a) = (\frac{x}{40} + \frac{y}{150} + a).$$

Clearly $\lambda_1(x, Gy, a) \leqslant \lambda_1(x, gy, a)$.

Similarly we can prove that

$$\lambda_n(Fx, y, a) \leqslant \lambda_n(fx, y, a), \lambda_n(x, Fy, a) \leqslant \lambda_n(x, fy, a)$$
$$\lambda_n(Gx, y, a) \leqslant \lambda_n(gx, y, a), \lambda_n(x, Gy, a) \leqslant \lambda_n(x, gy, a) \forall n = 2, 3, 4, ...7.$$

Consider

$$\begin{aligned}
&|S(Fx, Fx, Gy)| = \left|S(\tfrac{x}{16}, \tfrac{x}{16}, \tfrac{y}{12})\right| \\
&= |\tfrac{x}{16} - \tfrac{y}{12}| + i|\tfrac{x}{16} - \tfrac{y}{12}| = \tfrac{1}{4}[|\tfrac{x}{4} - \tfrac{y}{3}| + i|\tfrac{x}{4} - \tfrac{y}{3}|] \\
&< \tfrac{1}{3}[|\tfrac{x}{4} - \tfrac{y}{3}| + i|\tfrac{x}{4} - \tfrac{y}{3}|] \\
&\leqslant (\tfrac{x}{160} + \tfrac{y}{150} + \tfrac{1}{3})[|\tfrac{x}{4} - \tfrac{y}{3}| + i|\tfrac{x}{4} - \tfrac{y}{3}|] = \lambda_1(fx, gy, a)S(fx, fx, gy) \\
&\leqslant \lambda_1(fx, gy, a)S(fx, fx, gy) + \lambda_2(fx, gy, a)S(fx, fx, Fx) \\
&\quad + \lambda_3(fx, gy, a)S(gy, gy, Gy) + \lambda_4(fx, gy, a)[S(gy, gy, Fx) + S(fx, fx, Gy)] \\
&\quad + \lambda_5(fx, gy, a)\left(\tfrac{S(fx,fx,Fx)S(gy,gy,Gy)}{1+S(fx,fx,gy)}\right) \\
&\quad + \lambda_6(fx, gy, a)\left(\tfrac{S(gy,gy,Fx)S(fx,fx,Gy)}{1+S(fx,fx,gy)}\right) \\
&\quad + \lambda_7(fx, gy, a)\left(\tfrac{S(fx,fx,Fx)S(fx,fx,Gy)+S(gy,gy,Gy)S(gy,gy,Fx)}{1+S(fx,fx,Gy)+S(gy,gy,Fx)}\right).
\end{aligned}$$

Thus (2.2.5) is satisfied.

One can easily verify the remaining conditions of Theorem 2.2. Clearly $x = 0$ is the unique common fixed point of F, G, f and g.

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